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ON DE JONG'S CONJECTURE

BY

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ABSTRACT

Let X be a smooth projective curve over a finite field F_q. Let ρ be a continuous representation $\pi(X) \to GL_n(F)$, where $F = F_l((t))$ with F_l being another finite field of order prime to q.

Assume that $\rho|_{\pi(\bar{X})}$ is irreducible. De Jong's conjecture says that in this case $\rho(\pi(\bar{X}))$ is finite. As was shown in the original paper of de Jong, this conjecture follows from an existence of an F-valued automorphic form corresponding to ρ is the sense of Langlands. The latter follows, in turn, from a version of the Geometric Langlands conjecture.

In this paper we sketch a proof of the required version of the geometric conjecture, assuming that $char(F) \neq 2$, thereby proving de Jong's conjecture in this case.

1. Introduction

1.1. The purpose of this note is to indicate the proof of a partial case of de Jong's conjecture, proposed in [4]. Let us recall its formulation, combining Conjecture 2.3 and Theorem 2.17 of *loc. cit.*:

Let X be a smooth projective curve over a finite field $\mathbb{F}_q$, and let ρ be a continuous representation

$$\pi_1(X) \to GL_n(\mathbf{F}),$$

where $\mathbf{F} = \mathbb{F}_l((t))$ with $\mathbb{F}_l$ being another finite field of order coprime to q. Assume that $\rho|_{\pi_1(\overline{X})}$ is absolutely irreducible (here, as usual, $\pi_1(\overline{X}) \subset \pi_1(X)$ denotes the geometric fundamental group).

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CONJECTURE 1.2: *Under the above circumstances, $\rho(\pi_1(\overline{X}))$ is finite.*

In the same paper, de Jong showed that Conjecture 1.2 follows from a version of the Langlands conjecture with **F**-coefficients. (The corresponding version of the Langlands conjecture with $\overline{\mathbb{Q}_\ell}$-coefficients is now a theorem of Lafforgue.)

Namely, consider the double quotient $GL_n(K)\backslash GL_n(\mathbb{A})/GL_n(\mathbb{O})$, where K is the global field corresponding to X, $\mathbb{A}$ is the ring of adeles, and $\mathbb{O}$ is the subring of integral adeles. (Note that the above double quotient identifies with the set of isomorphism classes of rank n vector bundles on X, denoted $\mathrm{Bun}_n(\mathbb{F}_q)$.) For each place $x \in |X|$ one introduces the Hecke operators T^i_x, $i = 1, \ldots, n$ acting on the space of **F**-valued functions on $\mathrm{Bun}_n(\mathbb{F}_q)$.

Given a representation ρ: $\pi_1(X) \to GL_n(\mathbf{F})$, we say that a function f : $\mathrm{Bun}_n(\mathbb{F}_q) \to \mathbf{F}$ is a Hecke eigenform with eigenvalues corresponding to ρ, if for all x and i

$$T^i_x(f) = \lambda^i_x \cdot f,$$

with $\lambda^i_x = \mathrm{Tr}(\Lambda^i(\rho(Fr_x)))$, where $Fr_x \in \pi_1(X)$ is the Frobenius element corresponding to x (defined up to conjugacy), and Λ^i designates the i-th exterior power of the representation ρ.

In addition, in the space of all **F**-valued functions on $\mathrm{Bun}_n(\mathbb{F}_q)$ one singles out a subspace, stable under the operators T^i_x, of cuspidal functions. As in the $n = 2$ case, one shows that the subspace of cuspidal functions with a given central character is finite-dimensional.

Here is the relevant version of the Langlands conjecture:

CONJECTURE 1.3: *Let ρ be as in Conjecture 1.2. Then there exists a (non-zero) cuspidal Hecke-eigenform f_ρ with eigenvalues corresponding to ρ.*

By arguments of Sect. 4 of [4], one shows that Conjecture 1.3 implies Conjecture 1.2. In this paper we will be concerned with the proof of Conjecture 1.3.

1.4. Unfortunately, the proof of Conjecture 1.3 is not complete. Namely, we will have to rely on two pieces of mathematics that do not exist in the published literature.

One is the theory of étale sheaves with **F** coefficients, which should be parallel to the theory of $\mathbf{F}'$-adic sheaves, where $\mathbf{F}'$ is a local field of characteristic 0.

However, we do not expect that the construction of this theory is in any way different from its $\mathbf{F}'$-counterpart. Namely, we first consider étale sheaves of $\mathbb{F}_l[t]/t^i$-modules, and define the category of **O**-sheaves as the appropriate 2-projective limit, where **O** is the local ring $\mathbb{F}_l[[t]]$. We define the category of

$\mathbf{F}$-sheaves by inverting t, i.e., by quotienting out the category of $\mathbf{O}$-sheaves by t-torsion ones.

Thus, from now on we will assume the existence of such a theory, with formal properties analogous to that of $\mathbf{F}'$-sheaves. In particular, we assume the existence of the category $D(\mathcal{Y})$ of bounded complexes with constructible cohomology corresponding to an algebraic variety $\mathcal{Y}$ over $\mathbb{F}_q$, stable under the 6-functors. From this category one produces the abelian category of perverse sheaves, denoted $P(\mathcal{Y})$.

Finally, we must have the sheaf-function correspondence. In other words, given an object $\mathcal{K} \in D(\mathcal{Y})$, by taking traces of the Frobenius elements, we obtain an $\mathbf{F}$-valued function on $\mathcal{Y}(\mathbb{F}_q)$, and this operation is compatible with the operations of taking inverse image, direct image with compact supports and tensor product.

1.5. That said, our goal will be to prove a geometric version of Conjecture 1.3 (we refer the reader to Sect. 3.1 for the precise formulation of the latter).

Namely, to a representation ρ, as in Conjecture 1.2, we would like to associate an object $\mathcal{S}_\rho \in D(\mathrm{Bun}_n)$, where Bun_n denotes the moduli stack of rank n bundles on X, such that $\mathcal{S}_\rho$ is cuspidal and satisfies the Hecke eigencondition with respect to ρ. If such $\mathcal{S}_\rho$ exists, then by applying the sheaf-function correspondence, we obtain a function f_ρ on $\mathrm{Bun}_n(\mathbb{F}_q)$, which is a cuspidal Hecke eigenform with eigenvalues corresponding to ρ.

Our main result is Theorem 3.5, which says that if $l > 2n$, and ρ is as in Conjecture 1.2, then the object $\mathcal{S}_\rho \in D(\mathrm{Bun}_n)$ with the required properties exists.

In fact, we state and indicate the proof of a stronger result, namely, Theorem 3.6, which asserts the existence of $\mathcal{S}_\rho$ for any $l \neq 2$.

To summarize, this paper proves Conjecture 1.2 for $l \neq 2$ modulo the theory of $\mathbf{F}$-sheaves, and one more unpublished result, discussed below.

1.6. Even in order to formulate the geometric analog of Conjecture 1.3, i.e., Conjecture 3.2, one needs to rely on the realization of the category of representations of the Langlands dual group via spherical perverse sheaves on the affine Grassmannian.

This result was first announced by V. Ginzburg in the 1995's for perverse sheaves with coefficients in a field of characteristic zero. Recently, in [11], I. Mirković and K. Vilonen have established this result in a far greater generality (cf. Theorem 12.1 in *loc. cit*).

Namely, they work over a *ground field* of complex numbers, and consider sheaves in the analytic topology. They show that the category of spherical perverse sheaves with A-coefficients, where A is an arbitrary Noetherian commutative ring of finite cohomological dimension, is equivalent to the category of representations of the group-scheme $\check{G}_A$ on finitely generated A-modules, where $\check{G}_A$ is the split reductive group, whose root datum is dual to that of G.

For the purposes of this paper, however, we need an extension of the result of [11] to the case of an arbitrary ground field (in practice taken to be $\mathbb{F}_q$), and sheaves with **F**- and **O**-coefficients. From examining [11] it appears that the proof presented in *loc. cit.* carries over to this case. Therefore, we state the corresponding result as Theorem 2.2.

1.7. The geometric Langlands conjecture with coefficients of characteristic 0 has been proved in [5] and [8]. The proof of Theorem 3.5 follows verbatim the approach of *loc. cit.*, with one exception:

This exception is the discussion related to the notion of symmetric power of a local system (cf. Sect. 5), and this is the only original piece of work done in this paper.

Let us now describe the contents of the paper:

In Sect. 2 we recall the definition of affine Grassmannians and review the relevant versions of the geometric Satake equivalence, i.e., the realization of the Langlands dual group via spherical perverse sheaves, following [11]. We also introduce the Hecke stack, Hecke functors and the notion of Hecke eigensheaf.

Starting from Sect. 3 we restrict ourselves to the case $G = GL_n$. In the beginning of Sect. 3 we formulate several versions of the geometric Langlands conjecture for GL_n, and state the main result, Theorem 3.5, which amounts to proving one of the forms of the above conjecture when $\mathrm{char}(\mathbf{F}) > 2n$. We also formulate a stronger result, Theorem 3.6, which claims the ultimate form of the geometric Langlands conjecture when $\mathbf{F} \neq 2$.

In the rest of Sect. 3 we explain, following closely the exposition in [5], how Theorem 3.5 can be reduced to a certain vanishing statement, Theorem 4.2.

In Sect. 4 we formulate and prove Theorem 4.2, following [8]. The proof essentially amounts to introducing a certain quotient triangulated category $\widetilde{D}(\mathrm{Bun}_n)$ of $D(\mathrm{Bun}_n)$ and showing that the averaging functor $\mathrm{Av}^d_E : D(\mathrm{Bun}_n) \to D(\mathrm{Bun}_n)$ (whose vanishing for large d we are trying to prove) is well-defined and *exact* on the above quotient.

The proof of the latter fact essentially consists of two steps:

(1) Showing that the elementary functor $\mathrm{Av}^1_E : \widetilde{D}(\mathrm{Bun}_n) \to \widetilde{D}(\mathrm{Bun}_n)$ is exact.

(2) Showing that the exactness of Av^1_E implies the exactness of Av^d_E for any d.

Step (1) requires no modification compared to the case of characteristic 0 coefficients if $\mathrm{char}(\mathbf{F}) > 2n$ considered in [8], and a minor modification if we only assume $\mathrm{char}(\mathbf{F}) \neq 2$ (the latter case is treated in the Appendix).

Step (2) amounts to Proposition 4.6(2) and this is the only place in the paper that requires some substantial work. Essentially, Av^d_E is the d-th symmetric power (along X) of Av^1_E, and our goal is to express one through another. This is achieved by introducing a somewhat non-standard notion of external exterior power of a local system on a curve.

In Sect. 5 we first recall some notions from linear algebra, namely, the two versions of symmetric and exterior powers of a vector space and the corresponding Koszul complexes, when working over a field of positive characteristic.

We then review some basic properties of the construction of the (external) symmetric power of a local system on a curve, in particular, its behavior with respect to the perverse t-structure.

And finally, we introduce two versions of an external exterior power of a local system, and construct the corresponding external Koszul complex, which is used in the proof of Proposition 4.6(2) mentioned above.

In Appendix A, we indicate how, by refining some arguments of [8], one can relax the condition that $\mathrm{char}(\mathbf{F}) > 2n$ and treat the case of any $\mathbf{F}$ of characteristic different from 2.

Finally, in Appendix B we prove Theorem 2.6, which is a version of the geometric Satake equivalence over a symmetric power of the curve X.

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2. Review of the geometric Satake equivalence

From now on, we will work over an arbitrary algebraically closed ground field k of characteristic prime to l, and X will be a smooth projective curve $/k$.

In this section G will be an arbitrary reductive group over k. By $\check{G}$ we will denote its Langlands dual group, which we think of as a smooth group-scheme defined over $\mathbb{Z}$. We will denote by $\check{G}_{\mathbf{O}}$ the corresponding group-scheme over $\mathbf{O}$ and by $\check{G}_{\mathbf{F}}$ the corresponding reductive group over $\mathbf{F}$. We will denote by

$\mathrm{Rep}(\check{G}_{\mathbf{F}})$ the category of rational representations of $\check{G}_{\mathbf{F}}$ on finite-dimensional $\mathbf{F}$-vector spaces.

2.1. Let $x \in X$ be a point. If G is an algebraic group, let $\mathrm{Gr}_{G,x}$ denote the affine Grassmannian of G on X at x. In other words, $\mathrm{Gr}_{G,x}$ is the ind-scheme classifying the data of a G-bundle $\mathcal{P}_G$ on X with a trivialization β on $X - x$, i.e., $\mathcal{P}_G|_{X-x} \simeq \mathcal{P}^0_G|_{X-x}$, where $\mathcal{P}^0_G$ denotes the trivial G-bundle.

According to a theorem of Beauville and Laszlo, the data of $(\mathcal{P}_G, \beta)$ is equivalent to one, where instead of X we use the formal disc $\mathcal{D}_x$ around x, and instead of $X - x$ the formal punctured disc $\mathcal{D}^*_x$, cf. [7].

Let $G(\mathcal{K}_x)$ (resp., $G(\mathcal{O}_x)$) be the group ind-scheme (resp., group-scheme) classifying maps $\mathcal{D}^*_x \to G$ (resp., $\mathcal{D}_x \to G$). The description of $\mathrm{Gr}_{G,x}$ via $\mathcal{D}_x$ implies that the group ind-scheme $G(\mathcal{K}_x)$ acts on it by changing the data of β. The action of $G(\mathcal{O}_x)$ has the property that every finite-dimensional closed subscheme of $\mathrm{Gr}_{G,x}$ is contained in another finite-dimensional closed subscheme of $\mathrm{Gr}_{G,x}$, stable under the action of $G(\mathcal{O}_x)$. Therefore, the category $P^{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})$ of $G(\mathcal{O}_x)$-equivariant perverse sheaves (with $\mathbf{F}$-coefficients) on $\mathrm{Gr}_{G,x}$ makes sense.

The basic fact is that $P^{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})$ carries a natural structure of monoidal category. We refer to [7], where this is discussed in detail.

We will need the following result, which is a generalization of Theorem 12.1 of [11] to the case of an arbitrary ground field:

Theorem 2.2: *The monoidal structure on $P^{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})$ admits a natural symmetric commutativity constraint. The resulting tensor category is equivalent to the category* $\mathrm{Rep}(\check{G}_{\mathbf{F}})$.

Let $\mathrm{Aut}(\mathcal{D}_x)$ be the group-scheme of automorphisms of the formal disc $\mathcal{D}_x$. By functoriality, we have a natural action of $\mathrm{Aut}(\mathcal{D}_x)$ on $\mathrm{Gr}_{G,x}$. A part of Theorem 2.2 is the following statement (cf. Proposition 2.2 of [11]):

Corollary 2.3: *Every object of $P^{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})$ is $\mathrm{Aut}(\mathcal{D}_x)$-equivariant.*

Theorem 2.2 remains valid in the context of $\mathbf{O}$-coefficients. Namely, let us denote by $P^{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})_{\mathbf{O}}$ the category of $\mathbf{O}$-valued $G(\mathcal{O}_x)$-equivariant perverse sheaves on $\mathrm{Gr}_{G,x}$.* Then $P^{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})_{\mathbf{O}}$ is also a tensor category, equivalent

* Since $\mathbf{O}$ is a ring and not a field, Serre duality on the derived category of $\mathbf{O}$-modules does not preserve the t-structure. Therefore, the corresponding derived category of $\mathbf{O}$-sheaves (as in the $\mathbb{Z}_l$ case) possesses two natural perverse t-structures, interchanged by Verdier duality. Here we will use the “usual” one, for which the category of perverse sheaves is Noetherian.

to the category of representations of the group-scheme $\check{G}_{\mathbf{O}}$ on finitely generated $\mathbf{O}$-modules.

2.4. We will now discuss a version of Theorem 2.2, where instead of one point x we have several points moving along the curve X. For a positive integer d, let Gr^d_G be the symmetrized version of the Beilinson–Drinfeld Grassmannian living over $X^{(d)}$. Namely, a point of Gr^d_G is a triple $(D, \mathcal{P}_G, \beta)$, where D is an effective divisor of degree d on X, i.e., a point of $X^{(d)}$, $\mathcal{P}_G$ is a principal G-bundle on X, and β is a trivialization of $\mathcal{P}_G$ off the support of D.

The formal disc version of Gr^d_G can be spelled out as follows. Let S be a (test) scheme and let D_S be an S-point of $X^{(d)}$. Let Γ be the incidence divisor in $X \times X^{(d)}$, and let Γ_S be its pull-back to $X \times S$. Let $k \cdot \Gamma_S$ denote the k-th infinitesimal neighborhood of Γ_S and let $\widehat{\Gamma}_S$ be the completion of $X \times S$ along Γ_S. Note that it makes sense to speak about principal G-bundles on $\widehat{\Gamma}_S$, and of isomorphisms of two such bundles on $\widehat{\Gamma}^*_S := \widehat{\Gamma}_S - \Gamma_S$.

Thus, an S-point of Gr^d_G is a triple $(D_S, \mathcal{P}_G, \beta)$, where D_S is an S-point of $X^{(d)}$, $\mathcal{P}_G$ is a principal G-bundle on $\widehat{\Gamma}_S$, and β is an isomorphism $\mathcal{P}_G \simeq \mathcal{P}^0_G$ on $\widehat{\Gamma}^*_S$.

For a partition $\overline{d} : d = d_1 + \cdots + d_m$, let $sum_{\overline{d}}$ denote the natural morphism $X^{(d_1)} \times \cdots \times X^{(d_m)} \to X^{(d)}$. Let $X^{(\overline{d})}_{disj}$ denote the open subset of $X^{(d_1)} \times \cdots \times X^{(d_m)}$ corresponding to the condition that all the divisors $D_i \in X^{(d_i)}$, $i = 1, \ldots, m$ have pairwise disjoint supports. We have:

$$X^{(\overline{d})}_{disj} \underset{X^{(d)}}{\times} \mathrm{Gr}^d_G \simeq X^{(\overline{d})}_{disj} \underset{X^{(d_1)} \times \cdots \times X^{(d_m)}}{\times} (\mathrm{Gr}^{d_1}_G \times \cdots \times \mathrm{Gr}^{d_m}_G). \tag{1}$$

For an integer k, let $\mathcal{G}^d_k$ denote the group-scheme over $X^{(d)}$, whose S-points over a given $D_S \in \mathrm{Hom}(S, X^{(d)})$ is the group of maps $k \cdot \Gamma_S \to G$. Let $\mathcal{G}^d$ be the group-scheme $\underleftarrow{lim} \mathcal{G}^d_k$. We have a natural action of $\mathcal{G}^d$ on Gr^d_G. Note that the fiber of $\mathcal{G}^d$ at $D = \Sigma d_i \cdot x_i$ with x_i's distinct is $\Pi G(\mathcal{O}_{x_i})$. These isomorphisms are easily seen to be compatible with the factorization isomorphism of (1).

We let $P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$ denote the category of $\mathcal{G}^d$-equivariant perverse sheaves on Gr^d_G. We claim that for $d = d_1 + d_2$ there exists a natural convolution functor

$$\star\colon P^{\mathcal{G}^{d_1}}(\mathrm{Gr}^{d_1}_G) \times P^{\mathcal{G}^{d_2}}(\mathrm{Gr}^{d_2}_G) \to P^{\mathcal{G}^d}(\mathrm{Gr}^d_G). \tag{2}$$

Indeed, consider the ind-scheme $\mathrm{Conv}^{d_1,d_2}_G$ that classifies the data of

$$(D_1, D_2, \mathcal{P}^1_G, \beta^1, \mathcal{P}^G, \beta),$$

where $D_i \in X^{(d_i)}$, $\mathcal{P}^1_G, \mathcal{P}_G$ are principal G-bundles on X, $\beta_1 : \mathcal{P}^1_G|_{X - D_1} \simeq \mathcal{P}_G|_{X - D_1}$, $\beta : \mathcal{P}_G|_{X - D_2} \simeq \mathcal{P}^0_G|_{X - D_2}$. We can view $\mathrm{Conv}^{d_1,d_2}_G$ as a fibration over $\mathrm{Gr}_G^{d_2}$ with the typical fiber $\mathrm{Gr}_G^{d_1}$. Thus, for any perverse sheaf $\mathcal{T}_2$ on $\mathrm{Gr}_G^{d_2}$ and a $\mathcal{G}^{d_1}$-equivariant perverse sheaf $\mathcal{T}_1$ on $\mathrm{Gr}_G^{d_1}$ we can associate their twisted external product $\mathcal{T}_1\widetilde{\boxtimes}\mathcal{T}_2 \in P(\mathrm{Conv}_G^{d_1,d_2})$. We also have a natural projection p: $\mathrm{Conv}_G^{d_1,d_2} \to \mathrm{Gr}_G^d$ that sends a data $(D_1, D_2, \mathcal{P}_G^1, \beta^1, \mathcal{P}^G, \beta)$ as above to $(D_1 + D_2, \mathcal{P}_G^1, \beta \circ \beta^1)$. We set

$$\mathcal{T}_1 \star \mathcal{T}_2 := p_!(\mathcal{T}_1\widetilde{\boxtimes}\mathcal{T}_2).$$

The fact that the resulting object of the derived category is a perverse sheaf follows from the semi-smallness of convolution, cf. [11], Sect. 4.1. The fact that this perverse sheaf if $\mathcal{G}^d$-equivariant is evident, since all the objects involved in the construction, when viewed over $X^{(d)}$, carry a natural $\mathcal{G}^d$-action.

2.5. Note that if $\mathcal{C}$ is an **F**-linear abelian category, it makes sense to speak about objects of $\mathcal{C}$, endowed with an action of the algebraic group $\check{G}_{\mathbf{F}}$. We will take $\mathcal{C}$ to be the category of **F**-perverse sheaves on various schemes.

We introduce the category $P^{\check{G},d}$ to consist of perverse sheaves $\mathcal{K}$ on $X^{(d)}$, endowed with the following structure: For any ordered partition $\overline{d} : d = d_1 + \cdots + d_m$, the pull-back

$$\mathcal{K}^{\overline{d}} := sum_{\overline{d}}^*(\mathcal{K})|_{X^{(\overline{d})}_{disj}}$$

carries an action of $(\check{G}_{\mathbf{F}})^{\times m}$, such that the following two conditions hold:

1) If $\overline{d}' : d = d_1' + \cdots + d_{m'}'$ is a refinement of $\overline{d}$, then the isomorphism

$$\mathcal{K}^{\overline{d}}|_{X^{(\overline{d}')}_{disj}} \simeq \mathcal{K}^{\overline{d}'}$$

is compatible with the $(\check{G})^{\times m}$-actions via the diagonal map $(\check{G}_{\mathbf{F}})^{\times m} \to (\check{G}_{\mathbf{F}})^{\times m'}$.

2) If $\overline{d}' : d = d_1' + \cdots + d_{m'}'$ is obtained from $\overline{d}$ by a permutation, then the isomorphism of perverse sheaves induced by the isomorphism $X^{\overline{d}} \to X^{\overline{d}'}$ is compatible with the $(\check{G}_{\mathbf{F}})^{\times m}$-actions.

Note that for $d = d_1 + d_2$, the functor $\mathcal{K}_1, \mathcal{K}_2 \mapsto sum_{d_1,d_2}(\mathcal{K}_1 \boxtimes \mathcal{K}_2)$ gives rise to a functor

$$\star\colon P^{\check{G},d_1} \times P^{\check{G},d_2} \to P^{\check{G},d}. \tag{3}$$

The next result follows formally from Theorem 2.2:

THEOREM 2.6: *For every d there is a canonical equivalence of categories $P^{\mathcal{G}^d}(\mathrm{Gr}_G^d) \to P^{\check{G},d}$, compatible with the functors $\star$ of (2) and (3).*

The proof will be given in Appendix B.

We will denote by $P_{\mathbf{O}}^{\check{G},d}$ the corresponding version of $P^{\check{G},d}$ with $\mathbf{O}$-coefficients. Theorem 2.6 remains valid in this context as well, i.e., $P_{\mathbf{O}}^{\check{G},d}$ is equivalent to the category $P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)_{\mathbf{O}}$ of $\mathcal{G}$-equivariant perverse sheaves with $\mathbf{O}$-coefficients on Gr^d_G.

2.7. Let us take $d = 1$ and denote the corresponding ind-scheme Gr^1_G by $\mathrm{Gr}_{G,X}$, and the corresponding group-scheme $\mathcal{G}^1$ simply by $\mathcal{G}$. Note that $\mathrm{Gr}_{G,X}$ is just the relative over X version of $\mathrm{Gr}_{G,x}$, i.e., we have a projection $s\colon \mathrm{Gr}_{G,X} \to X$ and its fiber over $x \in X$ is $\mathrm{Gr}_{G,x}$.

It follows from Corollary 2.3, that to an object $V \in \mathrm{Rep}(\check{G}_{\mathbf{F}})$ one can canonically attach a perverse sheaf $\mathcal{T}_{V,X} \in P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$. In terms of the equivalence of Theorem 2.6, $\mathcal{T}_{V,X}$ corresponds to the constant sheaf $F_X \otimes V[1]$, as an object of $P^{\check{G},1}$.

Let Bun_G be the moduli stack of principal G-bundles on X. Let us recall the definition of the Hecke functors $H\colon \mathrm{Rep}(\check{G}_{\mathbf{F}}) \times D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G \times X)$.

Let $\mathcal{H}_{G,X}$ be the Hecke stack, i.e, the relative over Bun_G version of $\mathrm{Gr}_{G,X}$. More precisely, $\mathcal{H}_{G,X}$ classifies the data of $(x, \mathcal{P}_G, \mathcal{P}'_G, \beta)$, where $x \in X$, $\mathcal{P}_G, \mathcal{P}'_G$ are principal G-bundles on X, and β is an isomorphism $\mathcal{P}_G|_{X-x} \simeq \mathcal{P}'_G|_{X-x}$. We will denote by $\overleftarrow{h}$ (resp., $\overrightarrow{h}$) the natural map of stacks $\mathcal{H}_{G,X} \to \mathrm{Bun}_G$ that remembers the data of $\mathcal{P}'_G$ (resp., $\mathcal{P}_G$). We will view $\mathcal{H}_{G,X}$ as a fibration over Bun_G via $\overleftarrow{h}$, with the typical fiber $\mathrm{Gr}_{G,X}$:

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}} \mathcal{H}_{G,X} \xrightarrow{\overrightarrow{h}\times s} \mathrm{Bun}_G \times X.$$

Due to the $\mathcal{G}$-equivariance condition, to every $\mathcal{S} \in D(\mathrm{Bun}_G)$ and $\mathcal{T} \in P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$ we can associate their twisted external product $\mathcal{T}\widetilde{\boxtimes}\mathcal{S} \in D(\mathcal{H}_{G,X})$. We define the Hecke functor

$$H(V,\mathcal{S}) := (\overrightarrow{h} \times s)_!(\mathcal{T}_{V,X}\widetilde{\boxtimes}\mathcal{S}) \in D(\mathrm{Bun}_G \times X).$$

More generally, if $V_1, \ldots, V_d$ is a collection of objects of $\mathrm{Rep}(\check{G}_{\mathbf{F}})$, by iterating the above construction, for $\mathcal{S} \in D(\mathrm{Bun}_G)$ we obtain an object

$$H(V_1 \boxtimes \cdots \boxtimes V_d, \mathcal{S}) \in D(\mathrm{Bun}_G \times X^d).$$

As in [8], one shows that for any $\mathcal{S}$ as above, $H(V_1 \boxtimes \cdots \boxtimes V_d, \mathcal{S})$ is ULA with respect to the projection $\mathrm{Bun}_G \times X^d \to X^d$.

PROPOSITION 2.8: *Let V_1, V_2 be two objects of* $\mathrm{Rep}(\check{G}_{\mathbf{F}})$.

(1) *Let σ be the transposition acting on $X \times X$. We have a functorial isomorphism*

$$\sigma^*(H(V_1 \boxtimes V_2, \mathcal{S})) \simeq H(V_2 \boxtimes V_1, \mathcal{S}),$$

whose square is the identity map.

(2) *The restriction $H(V_1 \boxtimes V_2, \mathcal{S})|_{\mathrm{Bun}_G \times \Delta(X)}$ identifies canonically with $H(V_1 \otimes V_2, \mathcal{S})[1]$.*

The above proposition allows us to introduce Hecke eigensheaves. Let $E_{\check{G}}$ be a $\check{G}_{\mathbf{F}}$-local system on X, viewed as a tensor functor $V \mapsto E_{\check{G}}^V$ from $\mathrm{Rep}(\check{G}_{\mathbf{F}})$ to the category of $\mathbf{F}$-local systems on X.

We say that $\mathcal{S}_{E_{\check{G}}} \in D(\mathrm{Bun}_G)$ is a Hecke eigensheaf with respect to $E_{\check{G}}$ if we are given an isomorphism of functors $\mathrm{Rep}(\check{G}_{\mathbf{F}}) \to D(\mathrm{Bun}_G)$:

$$\alpha(V) : H(V, \mathcal{S}_{E_{\check{G}}}) \simeq \mathcal{S}_{E_{\check{G}}} \boxtimes E_{\check{G}}^V[1], \tag{4}$$

such that the conditions 1) and 2) below are satisfied.

Before formulating them, note that for any collection $V_1, \ldots, V_d$ of objects of $\mathrm{Rep}(\check{G}_{\mathbf{F}})$, by iterating $\alpha(\cdot)$ we obtain as isomorphism

$$H(V_1 \boxtimes \cdots \boxtimes V_d, \mathcal{S}_{E_{\check{G}}}) \simeq \mathcal{S}_{E_{\check{G}}} \boxtimes E_{\check{G}}^{V_1}[1] \boxtimes \cdots \boxtimes E_{\check{G}}^{V_m}[1].$$

We require that for $V_1, V_2 \in \mathrm{Rep}(\check{G}_{\mathbf{F}})$, the following two diagrams commute:

1)

$$\begin{array}{ccc} H(V_1 \boxtimes V_2, \mathcal{S}_{E_{\check{G}}}) & \longrightarrow & \mathcal{S}_{E_{\check{G}}} \boxtimes E_{\check{G}}^{V_1}[1] \boxtimes E_{\check{G}}^{V_2}[1] \\ \downarrow & & \downarrow \\ \sigma^*(H(V_2 \boxtimes V_1, \mathcal{S}_{E_{\check{G}}})) & \longrightarrow & \mathcal{S}_{E_{\check{G}}} \boxtimes \sigma^*(E_{\check{G}}^{V_2}[1] \boxtimes E_{\check{G}}^{V_1}[1]) \end{array}$$

2)

$$\begin{array}{ccc} H(V_1 \boxtimes V_2, \mathcal{S})|_{\mathrm{Bun}_G \times \Delta(X)} & \longrightarrow & \mathcal{S} \boxtimes E_{\check{G}}^{V_1}[1] \boxtimes E_{\check{G}}^{V_2}[1]|_{\mathrm{Bun}_G \times \Delta(X)} \\ \downarrow & & \downarrow \\ H(V_1 \otimes V_2, \mathcal{S})[1] & \longrightarrow & \mathcal{S} \boxtimes E_{\check{G}}^{V_1 \otimes V_2}[1] \end{array}$$

2.9. We need to introduce one more piece of notation related to the Hecke action.

Let $\mathcal{H}_G^d$ be the relative (over Bun_G) version of Gr_G^d. We have the diagram

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}} \mathcal{H}_G^d \xrightarrow{\overrightarrow{h} \times s} \mathrm{Bun}_G \times X^{(d)},$$

and we view $\mathcal{H}^d_G$ as a fibration over Bun_G via $\overleftarrow{h}$ with the typical fiber Gr^d_G.

As in Sect. 2.7, for an object $\mathcal{T} \in P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$ and $\mathcal{S} \in D(\mathrm{Bun}_G)$ we can form their twisted external product $\mathcal{T}\widetilde{\boxtimes}\mathcal{S} \in D(\mathcal{H}^d_G)$. We define

$$H(\mathcal{T},\mathcal{S}) := (\overrightarrow{h} \times s)_!(\mathcal{T}\widetilde{\boxtimes}\mathcal{S}) \in D(\mathrm{Bun}_G \times X^{(d)}).$$

We set $\mathcal{T}\star\mathcal{S} \in D(\mathrm{Bun}_G)$ to be the direct image of $H(\mathcal{T},\mathcal{S})$ under $\mathrm{Bun}_G \times X^{(d)} \to \mathrm{Bun}_G$.

By construction, for $\mathcal{T}_1 \in P^{\mathcal{G}^{d_1}}(\mathrm{Gr}^{d_1}_G)$ and $\mathcal{T}_2 \in P^{\mathcal{G}^{d_2}}(\mathrm{Gr}^{d_2}_G)$, we have a functorial isomorphism

$$(\mathcal{T}_1 \star \mathcal{T}_2) \star \mathcal{S} \simeq \mathcal{T}_1 \star (\mathcal{T}_2 \star \mathcal{S}).$$

We will denote by the same symbol $\star$ the resulting action of $P^{\check{G},d}$ on $D(\mathrm{Bun}_G)$.

3. Geometric Langlands conjecture

In this section we will work with **F**-sheaves, unless specified otherwise.

3.1. From now on we will specialize to the case $G = GL_n$. In this case we will denote Bun_G by Bun_n, and think of principal G-bundles as of rank n vector bundles on X.

Let V_0 denote the tautological n-dimensional representation of $\check{G} = GL_n$. A $\check{G}$-local system on X is the same thing as an n-dimensional local system, the correspondence being $E_{\check{G}} \mapsto E := E^{V_0}_{E_{\check{G}}}$.

Here is the formulation of the geometric Langlands conjecture:

Conjecture 3.2: *If E is an absolutely irreducible n-dimensional local system on X, then there exists a perverse sheaf $\mathcal{S}_E \in P(\mathrm{Bun}_n)$, which is a Hecke eigensheaf with respect to E. Moreover, $\mathcal{S}_E$ is cuspidal and irreducible on every connected component of* Bun_n.

If $\mathcal{S}_E \in D(\mathrm{Bun}_n)$ and E is an n-dimensional local system one can formulate a condition, weaker than the Hecke eigenproperty: We say that $\mathcal{S}_E$ is a weak Hecke eigensheaf with respect to E, if we are given isomorphisms $\alpha(\cdot)$ as in (4) for V of the form $\Lambda^i(V_0)$, $i = 1, \dots, n$.

The weak form of the geometric Langlands conjecture says that if E is an irreducible local system on X, then there exists a perverse sheaf $\mathcal{S}_E \in P(\mathrm{Bun}_n)$, which is a *weak* Hecke eigensheaf with respect to E, such that $\mathcal{S}_E$ is cuspidal and irreducible on every connected component of Bun_n.

Evidently, the existence of a weak Hecke eigensheaf corresponding to E is sufficient to guarantee the existence of an **F**-valued cuspidal Hecke eigenform corresponding to (the π_1-representation, corresponding to E) in the sense of Conjecture 1.3.

Yet another form of the Hecke eigencondition, this time specific to GL_n is as follows:

For an n-dimensional local system E, we say that $\mathcal{S}_E \in D(\mathrm{Bun}_n)$ has a GL_n-Hecke eigenproperty with respect to E if we are given an isomorphism $\alpha(\cdot)$ *only for* $V = V_0$, which satisfies condition 1) of the definition of Hecke eigensheaves for $V_1 \simeq V_2 \simeq V$.

Conjecture 3.3: *Let E be an arbitrary n-dimensional local system. Then if $\mathcal{S}_E \in D(\mathrm{Bun}_n)$ has a GL_n-Hecke eigenproperty with respect to E, then it satisfies, in fact, the full Hecke property.*

When $\mathrm{char}(\mathbf{F}) = 0$, the above conjecture was essentially proved in [6], using Springer correspondence. We do not have any real evidence in favor of this conjecture when $\mathrm{char}(\mathbf{F}) \neq 0$ (except in the case when $\mathcal{S}_E$ is cuspidal and perverse, which we consider in Appendix A). However, we have the following assertion:

Lemma 3.4: *Assume that* $\mathrm{char}(\mathbf{F}) > n$, *and let $\mathcal{S}_E \in D(\mathrm{Bun}_n)$ have a GL_n-Hecke eigenproperty with respect to E. Then $\mathcal{S}_E$ is a weak Hecke eigensheaf with respect to E.*

The proof of this lemma given in [5] in the case when $\mathrm{char}(\mathbf{F}) = 0$ is applicable here, since the proof relies on the semi-simplicity of representations of Σ_i on $\mathbf{F}$-vector spaces, which is valid if $i < \mathrm{char}(\mathbf{F})$.

We will prove the following:

Theorem 3.5: *Assume that* $\mathrm{char}(\mathbf{F}) > 2n$. *Then for every irreducible local system E on X there exists a (cuspidal, irreducible on every connected component) perverse sheaf $\mathcal{S}_E \in P(\mathrm{Bun}_n)$, which is a GL_n-Hecke eigensheaf with respect to E.*

Combined with Lemma 3.4 this proves the weak form of the geometric Langlands conjecture, and hence Conjecture 1.3, assuming that $\mathrm{char}(\mathbf{F}) > 2n$.

In fact, by relying on some more unpublished work, one can strengthen this result, and prove the following:

Theorem 3.6: *Assume that* $\mathrm{char}(\mathbf{F}) \neq 2$. *Then for an absolutely irreducible local system E on X there exists a (cuspidal, irreducible on every connected component) perverse sheaf $\mathcal{S}_E \in P(\mathrm{Bun}_n)$, which is a Hecke eigensheaf with respect to E.*

We will sketch the proof of this theorem in Appendix A. Of course, Theorem 3.6 implies de Jong's conjecture for any $l \neq 2$.

3.7. To prove Theorem 3.5 we will follow the strategy of [5]. Let Mod_n^d be the stack of upper modifications of length d; note that Mod_n^d is a closed substack of $\mathcal{H}^d_{GL_n}$ corresponding to the condition that the generic isomorphism of bundles $\beta\colon \mathcal{M} \to \mathcal{M}'$ is such that it extends to a regular map of coherent sheaves.

We set $\mathcal{T}_E^d$ to be the perverse sheaf on $\mathcal{H}^d_{GL_n}$ corresponding via the equivalence of Theorem 2.6 to the following object of P^{d,GL_n}:

$$(E \otimes V_0)^{(d)}[d]$$

(cf. Sect. 5.3 below, where symmetric powers of local systems are discussed). One easily shows that $\mathcal{T}_E^d$ is indeed supported on Mod_n^d.

Let Coh_0^d be the stack classifying torsion sheaves of length d. There is a natural smooth projection $\mathrm{Mod}_n^d \to \mathrm{Coh}_0^d$, and $\mathcal{T}_E^d$ is isomorphic (up to a shift) to the pull-back of a canonical perverse sheaf, denoted $\mathcal{L}_E^d$ on Coh_0^d, called the Laumon sheaf.

More explicitly, $\mathcal{L}_E^d$ is the Goresky–MacPherson extension of its own restriction to the open substack $\overset{\circ}{\mathrm{Coh}}{}_0^d$, corresponding to regular semi-simple torsion sheaves. This restriction is the pull-back of $\overset{\circ}{E}{}^{(d)}$ under the natural smooth morphism $\overset{\circ}{\mathrm{Coh}}{}_0^d \to \overset{\circ}{X}{}^{(d)}$.

We will need the following statement, whose char $= 0$ version was proved in [10]. For $d = d_1 + d_2$, let $\mathrm{Fl}_0^{d_1,d_2}$ denote the stack classifying short exact sequences $0 \to J_1 \to J \to J_2 \to 0$, where J_1 and J_2 are torsion coherent sheaves of lengths d_1 and d_2, respectively.

Let $\mathfrak{p}$ denote the natural projection $\mathrm{Fl}_0^{d_1,d_2} \to \mathrm{Coh}_0^d$, and let $\mathfrak{q}$ be the projection $\mathrm{Fl}_0^{d_1,d_2} \to \mathrm{Coh}_0^{d_1} \times \mathrm{Coh}_0^{d_2}$.

Theorem 3.8: *For any local system E we have:*

$$\mathfrak{q}_! \circ \mathfrak{p}^*(\mathcal{L}_E^d) \simeq \mathcal{L}_E^{d_1} \boxtimes \mathcal{L}_E^{d_2}.$$

3.9. Proof of Theorem 3.8. Consider first the open substack of $\mathrm{Coh}_0^{d_1} \times \mathrm{Coh}_0^{d_2}$ equal to $\mathfrak{q}(\mathfrak{p}^{-1}(\overset{\circ}{\mathrm{Coh}}{}_0^d))$. Over it both maps $\mathfrak{q}$ and $\mathfrak{p}$ are isomorphisms and the isomorphism stated in the theorem is evident. Therefore, we have to show that $\mathfrak{q}_! \circ p^*(\mathcal{L}_E^d)$ is a perverse sheaf, extended minimally from the above open substack.

The question being local, we can assume that E is trivial. Let us decompose E as a sum of 1-dimensional local systems $E = E_1 \oplus \cdots \oplus E_n$. For a partition

$\overline{d} : d = d^1 + \cdots + d^n$, let $\mathrm{Fl}_0^{\overline{d}}$ be the stack classifying flags of coherent torsion sheaves with successive quotients having lengths given by $\overline{d}$, and let $\mathfrak{p}^{\overline{d}}$ be the natural projection $\mathrm{Fl}_0^{\overline{d}} \to \mathrm{Coh}_0^d$. Let $\mathfrak{q}^{\overline{d}}$ denote the map $\mathrm{Fl}_0^{\overline{d}} \to \prod_i X^{(d^i)}$ obtained by taking supports of the successive quotients.

From Theorem 2.6 and Lemma 5.5, it follows that

$$\mathcal{L}_E^d \simeq \bigoplus_{\overline{d}} \mathfrak{p}_!^{\overline{d}} \circ (\mathfrak{q}^{\overline{d}})^*(E_1^{(d^1)} \boxtimes \cdots \boxtimes E_n^{(d^n)}).$$

Note that each $E_i^{(d^i)}$ is the constant sheaf on $X^{(d^i)}$, since E_1 is one-dimensional and trivial.

Thus, we have to compute

$$\mathfrak{q}_! \circ \mathfrak{p}^* \circ \mathfrak{p}_!^{\overline{d}} \circ (\mathfrak{q}^{\overline{d}})^*(\mathbf{F}_{\mathrm{Fl}_0^{\overline{d}}}).$$

Let Z denote the fiber product $\mathrm{Fl}_0^{\overline{d}} \underset{\mathrm{Coh}_0^d}{\times} \mathrm{Fl}_0^{d_1,d_2}$. It can be naturally subdivided into locally closed subvarieties (Schubert cells) numbered by the set

$$(\Sigma_{d_1} \times \Sigma_{d_2}) \backslash \Sigma_d / (\Sigma_{d^1} \times \cdots \times \Sigma_{d^n}),$$

or, which is the same, of the ways to partition $d_1 = d_1^1 + \cdots + d_1^n$, $d_2 = d_2^1 + \cdots + d_2^n$ with $d_1^i + d_2^i = d^i$. For each such pair of partitions, let us denote by $Z^{\overline{d}_1,\overline{d}_2}$ the corresponding subvariety in Z.

It is sufficient to show that the direct image of the constant sheaf on $Z^{\overline{d}_1,\overline{d}_2}$ under

$$Z^{\overline{d}_1,\overline{d}_2} \hookrightarrow Z \to \mathrm{Fl}_0^{d_1,d_2} \xrightarrow{\mathfrak{q}} \mathrm{Coh}_0^{d_1} \times \mathrm{Coh}_0^{d_2}$$

is a perverse sheaf, minimally extended from the open substack $\mathfrak{q}(\mathfrak{p}^{-1}(\overset{\circ}{\mathrm{Coh}}{}_0^d))$.

The above map factors as

$$Z^{\overline{d}_1,\overline{d}_2} \to \mathrm{Fl}_0^{\overline{d}_1} \times \mathrm{Fl}_0^{\overline{d}_1} \xrightarrow{\mathfrak{p}^{\overline{d}_1} \times \mathfrak{p}^{\overline{d}_2}} \mathrm{Coh}_0^{d_1} \times \mathrm{Coh}_0^{d_2},$$

where the first arrow is a generalized smooth fibration into affine spaces, of relative dimension 0. Therefore, the resulting object of $D(\mathrm{Coh}_0^{d_1} \times \mathrm{Coh}_0^{d_2})$ is isomorphic to $\mathfrak{p}_!^{\overline{d}_1}(\mathbf{F}_{\mathrm{Fl}_0^{\overline{d}_1}}) \boxtimes \mathfrak{p}_!^{\overline{d}_2}(\mathbf{F}_{\mathrm{Fl}_0^{\overline{d}_2}})$, and the latter is known (cf. Theorem 2.6) to be a perverse sheaf, minimally extended from the required open locus.

3.10. Proceeding as in [5], we produce from $\mathcal{T}_E^d$ the Whittaker sheaf, and subsequently an object $\mathcal{S}'_E \in D(\mathrm{Bun}'_n)$, where Bun'_n is the stack classifying pairs $(\mathcal{M} \in \mathrm{Bun}_n, \kappa : \Omega^{n-1} \to \mathcal{M})$.

Let $U \subset \mathrm{Bun}_n$ be the open substack, defined by the condition that $\mathcal{M} \in U$ if $\mathrm{Ext}^1(\Omega^{n-1}, \mathcal{M}) = 0$. Let U' be the preimage of U in Bun'_n. Clearly, the restriction of the projection π: $\mathrm{Bun}'_n \to \mathrm{Bun}_n$ to U is smooth.

We claim that the restriction of $\mathcal{S}'_E$ to U' is perverse and irreducible on every connected component. This follows as in [5], using Theorem 3.8 from the vanishing result, Theorem 4.2, discussed below.

Having established Theorem 4.2, and hence perversity and irreducibility of $\mathcal{S}'_E$, the next step is to show that $\mathcal{S}'_E$ descends to a perverse sheaf on Bun_n. We do it by the same argument involving Euler–Poincaré characteristics as in *loc. cit.* Namely, we have to show that the Euler–Poincaré characteristics of $\mathcal{S}'_E$ are constant along the fibers of the projection π.

As in [5], using Deligne's theorem, we show that the Euler–Poincaré characteristics of $\mathcal{S}'_E$ are independent of the local system. Therefore, it suffices to show that the Euler–Poincaré characteristics of $\mathcal{S}'_{E_0}$ are constant along the fibers of π, where E_0 is the *trivial* n-dimensional local system. When we work with $\mathbf{F}'$-sheaves (where $\mathbf{F}'$ is a local field of characteristic 0 with residue field $\mathbb{F}_l$), rather than with $\mathbf{F} = \mathbb{F}_l((t))$-sheaves, the corresponding fact follows from Sect. 6 of [5]. Therefore, it suffices to show that the Euler–Poincaré characteristics of $\mathcal{S}'_{E_0}$ at a given point of Bun'_n are the same in the $\mathbf{F}'$- and $\mathbf{F}$-situations. We will show this by comparing both sides with $\mathbb{F}_l$-sheaves.

Thus, let $\mathbf{F}_0$ be any of the local fields ($\mathbf{F}$ or $\mathbf{F}'$), and let $\mathbf{O}_0$ be the corresponding local ring. By a subscript ($\mathbf{F}_0$, $\mathbf{O}_0$ or $\mathbb{F}_l$) we will indicate which of the sheaf-theoretic contexts we are working in.

Consider the corresponding category $P^{d,GL_n}_{\mathbf{O}_0}$. We can form the object $(E_0 \otimes V_0)^{(d)}_{\mathbf{O}_0} \in P^{d,GL_n}_{\mathbf{O}_0}$, which under $\mathbf{O}_0 \to \mathbf{F_O}$ and $\mathbf{O}_0 \to \mathbb{F}_l$ specializes to $(E_0 \otimes V_0)^{(d)}_{\mathbf{F}_0}$ and $(E_0 \otimes V_0)^{(d)}_{\mathbb{F}_l}$, respectively. From Sect. 5.3 it follows that $(E_0 \otimes V_0)^{(d)}_{\mathbf{O}_0}$ is $\mathbf{O}_0$-flat.

Using Theorem 2.6 for $\mathbf{O}_0$, from $(E_0 \otimes V_0)^{(d)}_{\mathbf{O}_0}$ we produce the corresponding $\mathbf{O}_0$-perverse sheaf on Mod^d_n, which is also $\mathbf{O}_0$-flat. Finally, we produce the complex of $\mathbf{O}_0$-sheaves $(\mathcal{S}'_{E_0})_{\mathbf{O}_0}$ on Bun'_n. By the above flatness property

$$(\mathcal{S}'_{E_0})_{\mathbf{O}_0} \overset{L}{\underset{\mathbf{O}_0}{\bigotimes}} \mathbb{F}_l \simeq (\mathcal{S}'_{E_0})_{\mathbb{F}_l} \quad \text{and} \quad (\mathcal{S}'_{E_0})_{\mathbf{O}_0} \overset{L}{\underset{\mathbf{O}_0}{\bigotimes}} \mathbf{F}_0 \simeq (\mathcal{S}'_{E_0})_{\mathbf{F}_0},$$

where $(\mathcal{S}'_{E_0})_{\mathbb{F}_l}$ and $(\mathcal{S}'_{E_0})_{\mathbf{F}_0}$ are the corresponding complexes of $\mathbb{F}_l$-sheaves and $\mathbf{F}_0$-sheaves, respectively, on Bun'_n.

Let $\mathcal{K}_{\mathbf{O}_0}$ (resp., $\mathcal{K}_{\mathbb{F}_l}$, $\mathcal{K}_{\mathbf{F}_0}$) be the fiber of $(\mathcal{S}'_{E_0})_{\mathbf{O}_0}$ (resp., $(\mathcal{S}'_{E_0})_{\mathbb{F}_l}$, $(\mathcal{S}'_{E_0})_{\mathbf{F}_0}$) at a given point of Bun'_n. This is an object of the bounded derived category of finitely-generated $\mathbf{O}_0$-modules (resp., finite-dimensional $\mathbb{F}_l$- or $\mathbf{F}_0$-vector spaces), and we still have

$$\mathcal{K}_{\mathbf{O}_0} \overset{L}{\underset{\mathbf{O}_0}{\bigotimes}} \mathbb{F}_l \simeq \mathcal{K}_{\mathbb{F}_l} \quad \text{and} \quad \mathcal{K}_{\mathbf{O}_0} \overset{L}{\underset{\mathbf{O}_0}{\bigotimes}} \mathbf{F}_0 \simeq \mathcal{K}_{\mathbf{F}_0}.$$

Under these circumstances we always have: $\chi(\mathcal{K}_{\mathbf{F}_0}) = \chi(\mathcal{K}_{\mathbb{F}_l})$. Indeed, it suffices to consider separately the cases when $\mathcal{K}_{\mathbf{O}_0}$ is a flat finitely-generated $\mathbf{O}_0$-module (in which case the assertion is evident), or when $\mathcal{K}_{\mathbf{O}_0}$ is torsion. In the latter case, we can assume that $\mathcal{K}_{\mathbf{O}_0} \simeq \mathbb{F}_l$. Then $\mathcal{K}_{\mathbf{F}_0} = 0$, and $\mathcal{K}_{\mathbb{F}_l}$ has 1-dimensional cohomologies in degrees -1 and 0, i.e., $\chi(\mathcal{K}_{\mathbb{F}_l}) = 0$.

This proves the fact that $\mathcal{S}'_E$ descends to a perverse sheaf $\mathcal{S}_E$ on Bun_n. The fact that $\mathcal{S}_E$ satisfies the Hecke property follows from Theorem 3.8 as in [5], Sect. 8. The cuspidality of $\mathcal{S}_E$ also follows from the vanishing result, Theorem 4.2.

4. The vanishing result

4.1. To state Theorem 4.2 we have to recall the definition of the averaging functor $\mathrm{Av}^d_E : D(\mathrm{Bun}_n) \to D(\mathrm{Bun}_n)$. It is defined for $d \in \mathbb{N}$ and a local system E of an arbitrary rank.

By definition,

$$\mathrm{Av}^d_E(\mathcal{S}) = \overrightarrow{h}_!(\overleftarrow{h}^*(\mathcal{S}) \otimes \mathcal{T}^d_E).$$

In other words,

$$\mathrm{Av}^d_E(\mathcal{S}) = (E \otimes V_0)^{(d)}[d] \star \mathcal{S},$$

in the notation of Sect. 2.9.

The key step in the proof of perversity and irreducibility of $\mathcal{S}'_E$ on U' is the following:

THEOREM 4.2: *Let E be absolutely irreducible, of rank m with $m > n$, and d be $> (2g-2) \cdot m \cdot n$. Then the functor Av^d_E is identically equal to* 0.

4.3. To prove Theorem 4.2 we will have to analyze separately the cases of $\mathrm{char}(k) = 0$ and k of positive characteristic. Let us show that the former case reduces to the latter. (Of course, for de Jong's conjecture we need the case $k = \overline{\mathbb{F}}_q$.)

Indeed, if k is of characteristic 0, we can replace it by $\mathbb{C}$ and work with sheaves in the analytic topology. In this case, we can consider sheaves with coefficients in an arbitrary field F of characteristic l. If F is finite, the local system E is defined over a finitely generated sub-field of k, and the standard procedure reduces us to the case of a finite ground.

The case of a general F reduces to that of $\mathbb{F}_l$. Indeed, since the fundamental group of a curve is finitely generated, we can assume that our local system has coefficients in a ring A, finitely generated over $\mathbb{F}_l$. Then the vanishing of the functor Av^d_E over A would follow from the corresponding assertion at all geometric points of A, whose residue fields are finite.

4.4. Thus, for the rest of this section we will assume that the ground field k is of positive characteristic of order prime to l. We need this assumption in order to have the Artin–Schreier sheaf on the affine line, and the Fourier transform, which is used in the definition of quotient categories $\widetilde{D}(\mathrm{Bun}_n)$, see below.

The proof of Theorem 4.2 will follow the same lines as the proof of the analogous statement in the situation of char $= 0$ coefficients in [8]. We have:

THEOREM 4.5: *Under the assumptions of Theorem 4.2, the functor* Av^d_E *is exact in the sense of the perverse t-structure on* $D(\mathrm{Bun}_n)$.

We show as in [8], Appendix (or, alternatively, as in Sect. 2.1 of *loc. cit.*, which is slightly more cumbersome) that Theorem 4.5 implies Theorem 4.2. Thus, our goal from now on is to prove Theorem 4.5.

The first step is to introduce a quotient triangulated category $\widetilde{D}(\mathrm{Bun}_n)$ of $D(\mathrm{Bun}_n)$, which has Properties 0, 1, 2 of [8], Sect. 2.12. The construction of $\widetilde{D}(\mathrm{Bun}_n)$ and the verification of its properties given in Sect. 4–8 of *loc. cit.* goes through without modification in our situation.

The next step is to prove an analog of Theorem 2.14 of *loc. cit.*, which says that the functor Av^1_E is exact on the quotient category $\widetilde{D}(\mathrm{Bun}_n)$, provided that E is absolutely irreducible of rank strictly greater than n. Again, the argument presented in *loc. cit.* is applicable, since it only involves the action of symmetric groups Σ_k with $k \leq 2n$, and we have made the assumption $\mathrm{char}(\mathbf{F}) > 2n$.

The final step, which will require some substantial modifications in the case of coefficients of positive characteristic, is the following:

PROPOSITION 4.6: *Let* $\widetilde{D}(\mathrm{Bun}_n)$ *be a triangulated quotient category of* $D(\mathrm{Bun}_n)$*, satisfying Properties 0 and 1 above.*

1. *For any E, the functor* Av^d_E *descends to a well-defined functor on* $\widetilde{D}(\mathrm{Bun}_n)$.
2. *If E is such that the functor* Av^1_E *is exact on* $\widetilde{D}(\mathrm{Bun}_n)$*, then so is* Av^d_E *for any d.*

Once Proposition 4.6 is proved, we finish the proof of Theorem 4.5 as in Sect. 2.16 of [8].

4.7. Let $\widetilde{D}(\mathrm{Bun}_n \times S)$ be the system of quotient categories of $D(\mathrm{Bun}_n \times S)$, satisfying Properties 0 and 1. Recall the generalized Hecke functors $H(\cdot,\cdot)$ of Sect. 2.9. We will prove the following:

Proposition 4.8: *For any $\mathcal{T} \in P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$, the functor*

$$D(\mathrm{Bun}_n) \to D(\mathrm{Bun}_n \times X^{(d)})\colon \mathcal{S} \mapsto H(\mathcal{T},\mathcal{S})$$

descends to a well-defined exact functor $\widetilde{D}(\mathrm{Bun}_n) \to \widetilde{D}(\mathrm{Bun}_n \times X^{(d)})$.

Clearly, Proposition 4.8 implies Proposition 4.6(1). In addition, we have the following corollary:

Corollary 4.9: *Let $\mathcal{T}$ be an object of $P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$ supported over a subvariety of dimension $\leq i$ of $X^{(d)}$. Then the functor $\mathcal{S} \mapsto \mathcal{T} \star \mathcal{S}$ has the cohomological amplitude at most $[-i,i]$ on $\widetilde{D}(\mathrm{Bun}_n)$.*

The rest of this subsection is devoted to the proof of Proposition 4.8.

Lemma 4.10: *The functor*

$$H(V,\cdot)\colon D(\mathrm{Bun}_n) \to D(\mathrm{Bun}_n \times X)$$

(cf. Sect. 2.7) descends to a well-defined exact functor $\widetilde{D}(\mathrm{Bun}_n) \to \widetilde{D}(\mathrm{Bun}_n \times X)$ for any $V \in \mathrm{Rep}(\check{G}_{\mathbf{F}})$.

Proof: First, we claim that this is true for $V = \Lambda^i(V_0)$. This follows from the Springer correspondence (applicable here, since $\mathrm{char}(\mathbf{F}) > n$) as in Prop. 1.11 of [8]. The fact that $H(V,\cdot)$ is well-defined for any V follows now, since the classes of the representations of the form $\Lambda^i(V_0)$ generate the Grothendieck ring of $\mathrm{Rep}(\check{G}_{\mathbf{F}})$.

To prove the exactness assertion, it is enough to assume that V is irreducible. We will proceed by induction on the length of the highest weight of V. Since the statement is essentially Verdier self-dual, it is enough to prove that the functor $H(V,\cdot)\colon \widetilde{D}(\mathrm{Bun}_n) \to \widetilde{D}(\mathrm{Bun}_n \times X)$ is right-exact. However, for any such V, there exists a representation V' isomorphic to a tensor product of representations of the form $\Lambda^i(V_0)$, together with a surjection $V' \twoheadrightarrow V$, such that its kernel, V'', is an extension of irreducible representations with smaller highest weights.

For $\mathcal{S} \in \widetilde{P}(\mathrm{Bun}_n)$ we have a long exact cohomology sequence

$$\cdots \to h^i(H(V'', \mathcal{S})) \to h^i(H(V', \mathcal{S})) \to h^i(H(V, \mathcal{S})) \to h^{i+1}(H(V'', \mathcal{S})) \to \cdots$$

and by the induction hypothesis, we conclude that $H(V, \cdot)$ is right exact. ∎

Consider the diagonal stratification of $X^{(d)}$ numbered by the partitions $\overline{d}$: $d = d_1 + \cdots + d_k$. For each such partition consider the space

$$X_{\overline{d}} := (\underbrace{X \times \cdots \times X}_{k \text{ times}})_{disj},$$

which covers in a finite andétale way the corresponding stratum in $X^{(d)}$.

Let us denote by $\mathrm{Gr}_{G,\overline{d}}$ the fiber product $X_{\overline{d}} \underset{X^{(d)}}{\times} \mathrm{Gr}_G^d$. Note that $\mathrm{Gr}_{G,\overline{d}}$ is isomorphic to

$$\underbrace{\mathrm{Gr}_{G,X} \times \cdots \times \mathrm{Gr}_{G,X}}_{k \text{ times}} \underset{X^k}{\times} X_{\overline{d}}.$$

We can consider the group scheme $\mathcal{G}_{\overline{d}} := \mathcal{G}^{\times k}|_{X_{\overline{d}}}$, the category $P^{\mathcal{G}_{\overline{d}}}(\mathrm{Gr}_{G,\overline{d}})$ of $\mathcal{G}_{\overline{d}}$-equivariant perverse sheaves on $\mathrm{Gr}_{G,\overline{d}}$, and the corresponding Hecke functor

$$H(\cdot, \cdot) \colon P^{\mathcal{G}_{\overline{d}}}(\mathrm{Gr}_{G,\overline{d}}) \times D(\mathrm{Bun}_n) \to D(\mathrm{Bun}_n \times X_{\overline{d}}).$$

To prove the proposition, it is enough to show that the latter functor descends to a well-defined exact functor $P^{\mathcal{G}_{\overline{d}}}(\mathrm{Gr}_{G,\overline{d}}) \times \widetilde{D}(\mathrm{Bun}_n) \to \widetilde{D}(\mathrm{Bun}_n \times X_{\overline{d}})$.

Note that every irreducible object of $P^{\mathcal{G}_{\overline{d}}}(\mathrm{Gr}_{G,\overline{d}})$ has the form

$$(\mathcal{T}_{V_1,X} \boxtimes \cdots \boxtimes \mathcal{T}_{V_k,X}|_{X_{\overline{d}}}) \otimes \mathcal{K},$$

where $\mathcal{K}$ is an irreducible perverse sheaf on $X_{\overline{d}}$, and $V_1, \ldots, V_k$ are irreducible objects of $\mathrm{Rep}(\check{G}_{\mathbf{F}})$ (the notation $\mathcal{T}_{V,X}$ is as in Sect. 2.7).

For such an object of $P^{\mathcal{G}^d}(\mathrm{Gr}_G^d)$ the above functor $H(\cdot, \cdot)$ takes the form

$$\mathcal{S} \mapsto H(V_1 \boxtimes \cdots \boxtimes V_k, \mathcal{S})|_{X_{\overline{d}}} \otimes \mathcal{K}.$$

From Lemma 4.10 we obtain that this functor indeed descends to $\widetilde{D}(\mathrm{Bun}_n)$. The exactness statement also follows from Lemma 4.10, since any object of the form

$$H(V_1 \boxtimes \cdots \boxtimes V_k, \mathcal{S}) \in D(\mathrm{Bun}_n \times X^k)$$

is ULA with respect to the projection $\mathrm{Bun}_n \times X^k \to X^k$, cf. Lemma 3.7 of [8].

4.11. Now we are ready to prove Proposition 4.6(2). For that we need one more piece of preparatory material, namely, the notion of external exterior power of a local system. This notion is discussed in Sect. 5.6. Thus, for any integer d we have a perverse sheaf $\Lambda^{!,(d)}(E\otimes V_0)$ on $X^{(d)}$. Moreover, by Theorem 5.7(2), $\Lambda^{!,(d)}(E\otimes V_0)$ is naturally an object of P^{d,GL_n}.

According to Proposition 4.8, we have a well-defined functor

$$\star\colon P^{d,GL_n}\times \widetilde{D}(\mathrm{Bun}_n)\to \widetilde{D}(\mathrm{Bun}_n),$$

and we must prove the exactness of $(E\otimes V_0)^{(d)}\star$.

Assume that $\mathcal{S}$ belongs to $\widetilde{P}(\mathrm{Bun}_n)$. We will prove by induction that $(E\otimes V_0)^{(d)}\star\mathcal{S}$ also belongs to $\widetilde{P}(\mathrm{Bun}_n)$. Thus, we assume that the statement holds for $d'<d$. Since the situation is essentially Verdier self-dual, it is enough to show that $(E\otimes V_0)^{(d)}\star\mathcal{S}\in\widetilde{D}^{\leq 0}(\mathrm{Bun}_n)$.

Consider the complex

$$\begin{aligned}&\Lambda^{!,(d)}(E\otimes V_0)\to\Lambda^{!,(d-1)}(E\otimes V_0)\star(E\otimes V_0)\to\cdots\to\\&\Lambda^{!,(i)}(E\otimes V_0)\star(E\otimes V_0)^{(d-i)}\to\Lambda^{!,(i-1)}(E\otimes V_0)\star(E\otimes V_0)^{(d-i+1)}\to\cdots\to\\&(E\otimes V_0)\star(E\otimes V_0)^{(d-1)}\to(E\otimes V_0)^{(d)}\end{aligned}$$

of objects of P^{d,GL_n}, given by Theorem 5.7. Since this complex is exact, it is enough to show that each

$$(\Lambda^{!,(i)}(E\otimes V_0)\star(E\otimes V_0)^{(d-i)})\star\mathcal{S}$$

belongs to $\widetilde{D}^{\leq i-1}(\mathrm{Bun}_n)$ for $i=1,\ldots,d$.

By the induction hypothesis, we know that $(E\otimes V_0)^{(d-i)}\star\mathcal{S}\in\widetilde{D}^{\leq 0}(\mathrm{Bun}_n)$. Therefore, it suffices to show that the functor

$$\mathcal{S}\mapsto\Lambda^{!,(i)}(E\otimes V_0)\star\mathcal{S}$$

sends $\widetilde{D}^{\leq 0}(\mathrm{Bun}_n)$ to $\widetilde{D}^{\leq i-1}(\mathrm{Bun}_n)$

Note that by Corollary 4.9, for $\mathcal{S}\in\widetilde{D}^{\leq 0}(\mathrm{Bun}_n)$, the object $\Lambda^{!,(i)}(E\otimes V_0)\star\mathcal{S}$ does belong to $\widetilde{D}^{\leq i}(\mathrm{Bun}_n)$, since $X^{(i)}$ is i-dimensional. Therefore, it suffices to show that the top cohomology $h^i(\Lambda^{!,(i)}(E\otimes V_0)\star\mathcal{S})$ vanishes.

Consider the object $j_{!*}(\overset{\circ}{\Lambda}{}^{!,(i)}(E\otimes V_0))[i]\in P^{i,GL_n}$. We have an injective map

$$j_{!*}(\overset{\circ}{\Lambda}{}^{!,(i)}(E\otimes V_0)[i])\hookrightarrow\Lambda^{!,(i)}(E\otimes V_0),$$

and the cokernel is an object of P^{i,GL_n} supported on a subvariety of dimension $\leq i$. Therefore, by Corollary 4.9 and the long exact sequence, it suffices to show that

$$h^0(j_{!*}(\overset{\circ}{\Lambda}{}^{!,(i)}(E\otimes V_0)[i])\star\mathcal{S})=0$$

for $\mathcal{S} \in \widetilde{D}^{\leq 0}(\mathrm{Bun}_n)$. We have a surjection of perverse sheaves on $\overset{\circ}{X}{}^{(i)}$

$$j^*((\mathrm{sym}_i)_!((E \otimes V_0)^{\boxtimes i}))[i] \to \overset{\circ}{\Lambda}{}^{!,(i)}(E \otimes V_0)[i],$$

and hence also a surjection

$$(\mathrm{sym}_i)_!((E \otimes V_0)^{\boxtimes i})[i] \twoheadrightarrow j_{!*}(\overset{\circ}{\Lambda}{}^{!,(i)}(E \otimes V_0)[i])$$

of objects of P^{i,GL_n}. Again, by Corollary 4.9 and the long exact sequence, it suffices to show that

$$h^0((\mathrm{sym}_i)_!((E \otimes V_0)^{\boxtimes i})[i] \star \mathcal{S}) = 0$$

for $\mathcal{S} \in \widetilde{D}^{\leq 0}(\mathrm{Bun}_n)$. However,

$$(\mathrm{sym}_i)_!((E \otimes V_0)^{\boxtimes i})[i] \star \mathcal{S} = \underbrace{\widetilde{\mathrm{Av}}{}^1_E \circ \cdots \circ \widetilde{\mathrm{Av}}{}^1_E}_{i \text{ times}}(\mathcal{S})[i].$$

By the assumption, the functor $\widetilde{\mathrm{Av}}{}^1_E$ is exact. Hence, the above expression has zero cohomologies in all the degrees $> -i$ and, in particular, in degree 0, if $\mathcal{S} \in \widetilde{D}^{\leq 0}(\mathrm{Bun}_n)$.

5. Symmetric and exterior powers of local systems

In this section we will work with **F**-vector spaces, and **F**-local systems on X. However, the same results extend to flat **O**-modules and local systems.

5.1. Let V be a vector space over **F**. Let us denote by $\mathrm{Sym}^{!,2}(V)$ the subspace of $V^{\otimes 2}$ consisting of flip-invariant vectors. Note that $\mathrm{Sym}^{!,2}(V)$ is spanned by vectors of the form $v \otimes v$, $v \in V$. Let $\mathrm{Sym}^{!,d}(V)$ be the subspace of $V^{\otimes d}$ consisting of invariants of the symmetric group Σ_d. Of course, for $d = 1$, $\mathrm{Sym}^{!,1}(V) = V$ and for $d \geq 2$,

$$\mathrm{Sym}^{!,d}(V) = \bigcap_{1 \leq i \leq d-1} V^{\otimes i-1} \otimes \mathrm{Sym}^{!,2}(V) \otimes V^{\otimes d-1-i},$$

since Σ_d is generated by the simple reflections.

Let $\Lambda^{!,2}(V)$ be the subspace of $V^{\otimes 2}$ spanned by vectors of the form $v \otimes w - w \otimes v$, $v, w \in V$. For $d \geq 2$, let $\Lambda^{!,d}(V)$ be the subspace of $V^{\otimes d}$ equal to the intersection

$$\bigcap_{1 \leq i \leq d-1} V^{\otimes i-1} \otimes \Lambda^{!,2}(V) \otimes V^{\otimes d-1-i}.$$

Note that when $\mathrm{char}(\mathbf{F}) \neq 2$, we can define $\Lambda^{!,2}(V)$ as the subspace of σ-anti-invariants in $V^{\otimes 2}$, where σ is the transposition; hence in this case $\Lambda^{!,d}(V)$ coincides with the subspace of Σ_d-anti-invariants in $V^{\otimes d}$.

Let $\mathrm{Sym}^{*,d}(V)$ be the quotient space of $V^{\otimes d}$ by the sum of subspaces of the form $V^{\otimes i-1} \otimes \Lambda^{!,2}(V) \otimes V^{\otimes d-1-i}$ for $1 \leq i \leq d-1$. In other words, $\mathrm{Sym}^{*,d}(V)$ is the space of Σ_d-coinvariants in $V^{\otimes d}$.

Let $\Lambda^{*,d}(V)$ be the quotient of $V^{\otimes d}$ by the sum of subspaces of the form $V^{\otimes i-1} \otimes \mathrm{Sym}^{!,2}(V) \otimes V^{\otimes d-1-i}$ for $1 \leq i \leq d-1$.

Note that as representations of $GL(V)_{\mathbf{F}}$, $\Lambda^{*,d}(V)$ and $\Lambda^{!,d}(V)$ are irreducible and isomorphic to one another (but, of course, the isomorphism is not given by the map $\Lambda^{!,d}(V) \to V^{\otimes d} \to \Lambda^{*,d}(V)$, as the latter is zero if $\mathrm{char}(\mathbf{F})$ divides d). The isomorphism in question is induced by the endomorphism of $V^{\otimes d}$ given by $\mathbf{\Sigma}_{\sigma \in \Sigma_d} sign(\sigma) \cdot \sigma$. We will sometimes use the notation $\Lambda^i(V)$ to denote either of the above vector spaces.

By definition, $\Lambda^1(V) = V$ and $\Lambda^0(V) = \mathrm{Sym}^{*,0}(V) = \mathrm{Sym}^{!,0}(V) = \mathbf{F}$. If $d > \dim(V)$ one easily shows that $\Lambda^d(V) = 0$.

Lemma 5.2: *We have:*

(1) *If V is finite-dimensional, there are canonical isomorphisms*

$$\mathrm{Sym}^{*,d}(V^*) \simeq (\mathrm{Sym}^{!,d}(V))^* \quad \text{and} \quad \Lambda^{*,d}(V^*) \simeq (\Lambda^{!,d}(V))^*,$$

where * denotes the dual vector space.

(2) *If $V \simeq V_1 \oplus V_2$, there are canonical isomorphisms*

$$\mathrm{Sym}^{*,d}(V) \simeq \bigoplus_{d_1+d_2=d} \mathrm{Sym}^{*,d_1}(V_1) \otimes \mathrm{Sym}^{*,d_2}(V_2)$$

and

$$\Lambda^{*,d}(V) \simeq \bigoplus_{d_1+d_2=d} \Lambda^{*,d_1}(V_1) \otimes \Lambda^{*,d_2}(V_2),$$

and similarly for the !-versions.

(3) *The natural maps $\Lambda^{!,d}(V) \to \Lambda^{!,d-1}(V) \otimes V$ and $\mathrm{Sym}^{*,d-1}(V) \otimes V \to \mathrm{Sym}^{*,d}(V)$ give rise to the long exact sequence (called the Koszul complex)*

$$\Lambda^{!,d}(V) \to \Lambda^{!,d-1}(V) \otimes V \to \cdots \to \Lambda^{!,i}(V) \otimes \mathrm{Sym}^{*,d-i}(V) \to$$
$$\Lambda^{!,i-1}(V) \otimes \mathrm{Sym}^{*,d-i+1}(V) \to \cdots \to V \otimes \mathrm{Sym}^{*,d-1}(V) \to \mathrm{Sym}^{*,d}(V),$$

and the maps $\mathrm{Sym}^{!,d}(V) \to \mathrm{Sym}^{!,d-1}(V) \otimes V$ and $\Lambda^{*,d-1}(V) \otimes V \to \Lambda^{*,d}(V)$ give rise to the long exact sequence

$$\mathrm{Sym}^{!,d}(V) \to \mathrm{Sym}^{!,d-1}(V) \otimes V \to \cdots \to \mathrm{Sym}^{!,i}(V) \otimes \Lambda^{*,d-i}(V) \to$$
$$\mathrm{Sym}^{!,i-1}(V) \otimes \Lambda^{*,d-i+1}(V) \to \cdots \to V \otimes \Lambda^{*,d-1}(V) \to \Lambda^{*,d}(V).$$

Moreover, the above long exact sequences transform to one another under duality, if V is finite-dimensional.

Points (1) and (2) of this lemma are straightforward. For the proof of (3), we reduce the assertion to the case of $\dim(V) = 1$ using (2).

5.3. Let now X be a smooth curve, let $X^{(d)}$ denote its d-th symmetric power, i.e., $X^{(d)} = X^d/\Sigma_d$, and let sym_d denote the projection $X^d \to X^{(d)}$. It is well-known that $X^{(d)}$ is smooth and, in particular, the map sym_d is flat.

Let E be a local system on X; we are going to recall the construction of its d-th external symmetric power. Recall that sum_{d_1,d_2} denotes the addition morphism $X^{(d_1)} \times X^{(d_2)} \to X^{(d)}$. For $\mathcal{S}_1 \in D(X^{(d_1)})$, $\mathcal{S}_2 \in D(X^{(d_2)})$, let $\mathcal{S}_1 \star \mathcal{S}_2 \in D(X^{(d)})$ denote the object $(sum_{d_1,d_2})_!(\mathcal{S}_1 \boxtimes \mathcal{S}_2)$.

Let $E^{\boxtimes d} \in D(X^d)$ denote the external power of E; this sheaf is naturally Σ_d-equivariant. Consider

$$E \star \cdots \star E \star \cdots \star E \simeq (\mathrm{sym}_d)_!(E^{\boxtimes d}) \in D(X^{(d)}).$$

Since the map sym_d is Σ_d-invariant, this sheaf is Σ_d-equivariant.

Recall that if Σ is a finite group (acting trivially on a variety $\mathcal{Y}$), we have the derived functor of invariants

$$R\,\mathrm{Inv}_\Sigma \colon D^+(\mathcal{Y})^\Sigma \to D^+(\mathcal{Y})^\cdot.$$

By applying this functor to $(\mathrm{sym}_d)_!(E^{\boxtimes d})$ we obtain an object

$$R\,\mathrm{Inv}_{\Sigma_d}((\mathrm{sym}_d)_!(E^{\boxtimes d})) \in D^+(X^{(d)}).$$

Finally, we set $E^{(d)}$ to be the 0-th cohomology *in the usual t-structure* of the complex $R\,\mathrm{Inv}_{\Sigma_d}((\mathrm{sym}_d)_!(E^{\boxtimes d}))$. In other words, $E^{(d)}$ is obtained from $(\mathrm{sym}_d)_!(E^{\boxtimes d})$ by taking non-derived Σ_d-invariants in *the abelian category of sheaves.*

Since the functor of stalks is exact on the category of sheaves, we obtain that for a point $\imath_D\colon \mathrm{pt} \to X^{(d)}$, where $D = \Sigma d_i \cdot x_i$ is an effective divisor of degree d with the x_i's pairwise distinct,

$$\imath_D^*(E^{(d)}) \simeq \bigotimes_i Sym^{!,d_i}(E_{x_i}),$$

where E_{x_i} denotes the stalk of E at x_i.

For two positive integers d_1, d_2 recall the subset

$$(X^{(d_1)} \times X^{(d_2)})_{disj} \subset X^{(d_1)} \times X^{(d_2)}.$$

We have:

$$(5)\qquad sum^*_{d_1,d_2}(E^{(d)})|_{(X^{(d_1)}\times X^{(d_2)})_{disj}} \simeq (E_1^{(d_1)} \boxtimes E_2^{(d_2)})|_{(X^{(d_1)}\times X^{(d_2)})_{disj}}.$$

Let $j\colon \overset{\circ}{X}{}^{(d)} \to X^{(d)}$ denote the embedding of the complement to the diagonal divisor. It is easy to see that $\overset{\circ}{E}{}^{(d)} := E^{(d)}|_{\overset{\circ}{X}{}^{(d)}}$ is a local system.

Proposition 5.4: *The complex $E^{(d)}[d]$ is a perverse sheaf; moreover,*

$$E^{(d)}[d] \simeq j_{!*}(\overset{\circ}{E}{}^{(d)}[d]).$$

Note that the proposition implies that the construction of $E^{(d)}$ is essentially Verdier self-dual, i.e., $\mathbb{D}(E^{(d)}[d]) \simeq (E^*)^{(d)}[d]$, where E^* is the dual local system. This is so, because the isomorphism obviously holds over $\overset{\circ}{X}{}^{(d)}$, from which both sides are extended minimally. In particular, we have a canonical projection $(\mathrm{sym}_d)_!(E^{\boxtimes d}) \to E^{(d)}$.

The above fact about self-duality implies the following description of the co-stalks of $E^{(d)}$: For $D = \Sigma d_i \cdot x_i$ as above,

$$i_D^!(E^{(d)}) \simeq \bigotimes_i Sym^{*,d_i}(E_{x_i})[-2d].$$

Note also that from the proposition it follows that $E^{(d)}[d]$ embeds into the perverse sheaf $h^0_{perv}(R\,\mathrm{Inv}_{\Sigma_d}((\mathrm{sym}_d)_!(E^{\boxtimes d}))[d])$, where h^0_{perv} denotes the functor of taking the 0-th cohomology in the perverse t-structure. However, this map is not in general an isomorphism.

Indeed, let $\mathbf{F}$ be of characteristic 2, take $d = 2$ and E to be the trivial 1-dimensional local system. Then, $(\mathrm{sym}_2)_!(E^{\boxtimes 2}) \simeq (\mathrm{sym}_d)_!(\mathbf{F}_{X^2})$, and we have a canonical embedding $\mathbf{F}_{X^{(2)}} \to (\mathrm{sym}_d)_!(\mathbf{F}_{X^2})$, whose cone is $j_!(\mathbf{F}_{\overset{\circ}{X}{}^{(2)}})$. By duality, we have an embedding of perverse sheaves $j_*(\mathbf{F}_{\overset{\circ}{X}{}^{(2)}})[2] \to (\mathrm{sym}_d)_!(\mathbf{F}_{X^2})[2]$, and it is easy to see that $j_*(\mathbf{F}_{\overset{\circ}{X}{}^{(2)}})[2]$ identifies with the subobject of Σ_2-invariants in $(\mathrm{sym}_d)_!(\mathbf{F}_{X^2})[2]$.

To prove Proposition 5.4 we will need the following lemma, which follows from Lemma 5.2:

Lemma 5.5: *For $E = E_1 \oplus E_2$ we have a canonical isomorphism:*

$$E^{(d)} \simeq \bigoplus_{d_1+d_2=d} E_1^{(d_1)} \star E_2^{(d_2)}.$$

Proof of Proposition 5.4: The question being étale-local, we can assume that the local system E is trivial. Hence we can decompose $E = E_1 \oplus \cdots \oplus E_m$, where the E_i's are 1-dimensional. By Lemma 5.5, we have:

$$E^{(d)} \simeq \bigoplus_{\overline{d}} E_1^{(d_1)} \star \cdots \star E_m^{(d_m)},$$

where $\overline{d} = (d_1, \ldots, d_m)$ runs over the set of m-tuples of non-negative integers with $\Sigma_i d_i = d$.

It is easy to see that for the trivial 1-dimensional local system its d-th symmetric power is the trivial 1-dimensional local system on $X^{(d)}$. This implies that assertion of the proposition, since for each $\overline{d}$, the map $X^{(d_1)} \times \cdots \times X^{(d_m)} \to X^{(d)}$ is finite. ∎

5.6. We will now construct another complex of sheaves, denoted $\Lambda^{!,(d)}(E)$, on $X^{(d)}$, called the external exterior power of E.

Theorem 5.7: *For every $d \geq 1$ there exists a canonically defined complex $\Lambda^{!,(d)}(E) \in D(X^{(d)})$ endowed with a map α_d: $\Lambda^{!,(d)}(E) \to \Lambda^{!,(d-1)}(E) \star E$, such that:*

1. *The restriction $\overset{\circ}{\Lambda}{}^{!,(d)}(E) := j^*(\Lambda^{!,(d)}(E))$ is a local system and*
$$\overset{\circ}{\Lambda}{}^{!,(d)}(E) \simeq R\,\mathrm{Inv}_{\Sigma_d}(j^*((\mathrm{sym}_d)_!(E^{\boxtimes d})) \otimes sign).$$
2. *For $d_1 + d_2 = d$ the restriction of $sum^*_{d_1,d_2}(\Lambda^{!,(d)}(E))$ to $(X^{(d_1)} \times X^{(d_2)})_{disj}$ is canonically isomorphic to the restriction to this open subset of $\Lambda^{!,(d_1)}(E) \boxtimes \Lambda^{!,(d_2)}(E)$.*
3. *$\Lambda^{!,(d)}(E)[d]$ is perverse.*
4. *The composition*
$$\Lambda^{!,(d)}(E) \xrightarrow{\alpha_d} \Lambda^{!,(d-1)}(E) \star E \xrightarrow{\alpha_{d-1}} \Lambda^{!,(d-2)}(E) \star E \star E \to \Lambda^{!,(d-2)}(E) \star E^{(2)}$$
is zero and the resulting complex of perverse sheaves
$$\Lambda^{!,(d)}(E)[d] \to \Lambda^{!,(d-1)}(E) \star E[d] \to \cdots \to \Lambda^{!,(i)}(E) \star E^{(d-i)}[d] \to$$
$$\Lambda^{!,(i-1)}(E) \star E^{(d-i+1)}[d] \to \cdots \to E \star E^{(d-1)}[d] \to E^{(d)}[d]$$
is exact.
5. *For a divisor $D = \Sigma d_i \cdot x_i$ with the x_i's pairwise distinct, the co-stalk $i^!_D(\Lambda^{!,(d)}(E))$ is (quasi-) isomorphic to $\bigotimes_i \Lambda^{!,d_i}(E_{x_i})[-2d]$, so that the co-stalk of the complex of point (4) identifies with the product over i of the Koszul complexes of Lemma 5.2(3).*

Note that the construction of $\Lambda^{!,(d)}(E)$ is not Verdier self-dual. Set

$$\Lambda^{*,(d)}(E) := \mathbb{D}(\Lambda^{!,(d)}(E^*))[-2d].$$

Then $\Lambda^{*,(d)}(E)$ would satisfy the same properties (1), (2) and (3) of Theorem 5.7 as $\Lambda^{!,(d)}(E)$. Instead of point (4) we will have an exact complex

$$E^{(d)}[d] \to E^{(d-1)} \star E[d] \to \cdots \to E^{(i)} \star \Lambda^{*,(d-i)}(E)[d] \to$$
$$E^{(i-1)} \star \Lambda^{*,(d-i+1)}(E)[d] \to \cdots \to E \star \Lambda^{*,(d-1)}(E)[d] \to \Lambda^{*,(d)}(E)[d].$$

Instead of point (5), we would be able to describe the stalks of $\Lambda^{*,(d)}(E)$:

$$i_D^*(\Lambda^{*,(d)}(E)) \simeq \bigotimes_i \Lambda^{*,d_i}(E_{x_i}).$$

Observe also that when $\mathrm{char}(\mathbf{F}) = 0$, both $\Lambda^{!,(d)}(E)$ and $\Lambda^{*,(d)}(E)$ are isomorphic to the minimal extension $j_{!*}(\overset{\circ}{\Lambda}{}^{!,(d)}(E))$.

5.8. Proof of Theorem 5.7. We proceed by induction on d. Evidently, for $d = 1$ we can take $\Lambda^{!,(d)}(E) = E \in D(X)$. Thus, we can assume that $\Lambda^{!,(i)}(E)$ satisfying conditions (1)–(5) of Theorem 5.7 have been constructed for $i < d$.

Define $\Lambda^{!,(d)}(E) \in D(X^{(d)})[d]$ to be represented by the complex *of perverse sheaves*

$$\mathcal{K}(d,E) := \Lambda^{!,(d-1)}(E) \star E[d] \to \cdots \to \Lambda^{!,(i)}(E) \star E^{(d-i)}[d]$$
$$\to \cdots \to E \star E^{(d-1)}[d] \to E^{(d)}[d].$$

(Here we are using the fact that the category of complexes of perverse sheaves maps to $D(X^{(d)})$; in fact, due to a theorem of Beilinson, the corresponding functor from the derived category of perverse sheaves to $D(X^{(d)})$ is an equivalence.)

It is easy to see that $\Lambda^{!,(d)}(E)$ satisfies conditions (1) and (2). Let us show that $\Lambda^{!,(d)}(E)[d]$ is a perverse sheaf. Since the question is étale-local, we can assume that E is the trivial local system; and let us write $E = E_1 \oplus \cdots \oplus E_m$, where the E_i's are trivial 1-dimensional.

Lemma 5.9: *For $E = E_1 \oplus E_2$ we have a canonical isomorphism:*

$$\Lambda^{!,(d)}(E) \simeq \bigoplus_{d_1+d_2=d} \Lambda^{!,(d_1)}(E_1) \star \Lambda^{!,(d_2)}(E_2).$$

Proof: Assume the validity of the lemma for $d' < d$. Then, by the induction hypothesis and Lemma 5.5, we obtain an isomorphism of complexes of perverse sheaves:

$$\mathcal{K}(d,E) \simeq \bigoplus_{d_1+d_2=d} \mathcal{K}(d_1,E_1) \star \mathcal{K}(d_2,E_2).$$

This implies our assertion. ∎

Thus, by decomposing E as a direct sum $E = E_1 \oplus \cdots \oplus E_m$ with the E_i's being 1-dimensional, we reduce the perversity assertion to the case when E is itself 1-dimensional. In the latter case we claim that

$$\Lambda^{!,(d)}(E) \simeq j_*(\overset{\circ}{\Lambda}{}^{!,(d)}(E)).$$

Indeed, by point (5) and the induction hypothesis, the co-stalk of $\Lambda^{!,(d)}(E)$ at $D = \Sigma d_i \cdot x_i$ is quasi-isomorphic to

$$\bigotimes_i \Lambda^{!,d_i}(E_{x_i})[-2d].$$

Therefore, if one of the d_i's is > 1, then the corresponding $\Lambda^{!,d_i}(E_{x_i}) = 0$, and the above expression vanishes. Moreover, this shows that the constructed complex $\Lambda^{!,(d)}(E)$ satisfies condition (5) of the theorem.

Hence, the perversity assertion follows from the fact that the embedding $j\colon \overset{\circ}{X}{}^{(d)} \to X^{(d)}$ is affine.

Note that the last point of the proof shows $\Lambda^{!,(d)}(E)$ is not a sheaf in the usual t-structure (cf. example preceding the proof of Proposition 5.4). However, the object $\Lambda^{*,(d)}(E)$ is always a sheaf.

6. Appendix A: Proof of Theorem 3.6

6.1. The assertion of Theorem 3.6 divides into two parts. First, we will show that Theorem 3.5 remains valid under the assumption that $\mathrm{char}(\mathbf{F}) \neq 2$. Secondly, we will prove a particular case of Conjecture 3.3 under the assumption that $\mathcal{S}_E$ is a cuspidal perverse sheaf.

The ingredient in the proof of Theorem 3.5 that relied on the assumption that $\mathrm{char}(\mathbf{F}) > 2n$ was the proof of Theorem 4.2. The latter used this assumption in the following two places:

1) Proof of the fact that if E is irreducible of rank $> n$, then the functor Av^1_E is exact on $\widetilde{D}(\mathrm{Bun}_n)$.

2) Proof of Lemma 4.10.

Let us first treat point 2). Let us call a representation $V \in \mathrm{Rep}(\check{G}_{\mathbf{F}})$ positive if the action of the group GL_n on it extends to an action of the semi-group $\mathrm{Mat}_{n,n}$. Every positive V can be decomposed into a direct sum of V_d, $d \geq 0$, according to the action of the center. Equivalently, V is positive of degree d if it can be realized as a quotient of (several copies of) $V_0^{\otimes d}$.

Let $\mathrm{Gr}^+_{GL_n,x} \subset \mathrm{Gr}_{G,x}$ be the "positive" part of the affine Grassmannian, corresponding to the condition that the modification of vector bundles

β: $\mathcal{M} \to \mathcal{M}^0$ is such that $\mathcal{M}^0 \to \mathcal{M}$ is a regular map of coherent sheaves. One can show that V is positive if and only if the corresponding perverse sheaf on $\mathrm{Gr}_{G,x}$ is supported on $\mathrm{Gr}^+_{GL_n,x}$.

Clearly, if V is any finite-dimensional representation of GL_n, by multiplying it by a sufficiently high power of the determinant, we can make it positive.

Therefore, to prove Lemma 4.10 it suffices to show that the functor $H(V,\cdot)$ is well-defined and exact on $\widetilde{D}(\mathrm{Bun}_n)$ for V positive. However, the proof of the latter fact can be obtained by the same argument as the proof of this statement for $V = V_0$ in [8], Sect. 7.

6.2. Let us now treat point 1) above. The assumption on $\mathrm{char}(\mathbf{F})$ was used in the proof of Theorem 2.14 of [8] in the following situation:

Recall that if Σ is a finite group, we have the abelian category $\widetilde{P}^{\Sigma}(\mathrm{Bun}_n \times X)$, which is a Serre quotient of the category of Σ-equivariant perverse sheaves on $\mathrm{Bun}_n \times X$, with the group Σ acting trivially.

We will take $\Sigma = \Sigma_i$, and consider the functor

$$\mathcal{S} \mapsto (\mathcal{S} \otimes sign)_{\Sigma_i}$$

of Σ_i-anti-coinvariants.

This functor is right-exact on $P^{\Sigma_i}(\mathrm{Bun}_n \times X)$, and hence also on $\widetilde{P}^{\Sigma}(\mathrm{Bun}_n \times X)$, and we need to replace by it the (exact, because of the assumption on the characteristic) functor $\mathcal{S} \mapsto \mathrm{Hom}_{\Sigma_i}(sign, \mathcal{S})$ considered in Sect. 3 of [8].

To make the argument work, we need to insure that an analog of Proposition 1.11 of [8] holds in our situation. Namely, let us consider the functor $\widetilde{P}(\mathrm{Bun}_n \times S) \to \widetilde{P}^{\Sigma_i}(\mathrm{Bun}_n \times X \times S)$ given by

$$\mathcal{S} \mapsto H_S^{\boxtimes i}(\mathcal{S})|_{\mathrm{Bun}_n \times \Delta(X) \times S}[1-i].$$

This functor maps to the abelian category because of Property 1 of $\widetilde{D}(\mathrm{Bun}_n)$ and the ULA assertion (cf. Lemma 3.7 of [8]). We have:

PROPOSITION 6.3: *If* $\mathrm{char}(\mathbf{F}) \neq 2$, *then the right-exact functor*

$$\mathcal{S} \mapsto ((H_S^{\boxtimes i}(\mathcal{S})|_{\mathrm{Bun}_n \times \Delta(X) \times S}[1-i]) \otimes sign)_{\Sigma_i}$$

mapping $\widetilde{P}(\mathrm{Bun}_n \times S)$ *to* $\widetilde{P}(\mathrm{Bun}_n \times X \times S)$ *is* 0 *if* $i > n$, *and is isomorphic to* $\mathcal{S} \mapsto m^*(\mathcal{S})[1]$ *if* $i = n$.

Proof: Using Lemma 4.10, the functor considered in the proposition is isomorphic to

$$\mathcal{S} \mapsto H((V_0 \otimes sign)^{\otimes i}_{\Sigma_i}, \mathcal{S}).$$

Hence, the assertion follows the fact that if $\mathrm{char}(\mathbf{F}) \neq 2$, then $(V_0 \otimes sign)^{\otimes i}_{\Sigma_i} \simeq \Lambda^i(V_0)$. ∎

6.4. Finally, let us show that if $\mathcal{S}_E$ is a cuspidal perverse sheaf, which has a GL_n-Hecke property with respect to a local system E, then, in fact, it satisfies the full Hecke property.

First, note that it suffices to construct the functorial isomorphisms $\alpha(V)$ of (4), and verify their properties, for representations V, which are positive.

We begin with the following observation:

LEMMA 6.5: *For any $V \in \mathrm{Rep}(\check{G}_{\mathbf{F}})$, and $\mathcal{S} \in D(\mathrm{Bun}_n)$, which is cuspidal and perverse, the object $H(V, \mathcal{S})$ is a perverse sheaf.*

Proof: Since the assertion is essentially Verdier self-dual, it suffices to show that $H(V, \mathcal{S})$ belongs to $D^{\leq 0}(\mathrm{Bun}_n \times X)$.

Suppose the contrary, and consider the truncation map

$$H(V, \mathcal{S}) \to \tau^{>0}(H(V, \mathcal{S})). \tag{6}$$

By Lemma 4.10, $H(V, \mathcal{S})$ is exact on $\widetilde{D}(\mathrm{Bun}_n \times X)$. By assumption, $\mathcal{S}$ is cuspidal, which implies that $H(V, \mathcal{S})$ is also cuspidal.

Property 2 of $\widetilde{D}(\mathrm{Bun}_n \times X)$ (cf. [8], Sect. 2.12) implies that the truncation map (6) is 0, which is a contradiction. ∎

Let $\underline{V_0^*}$ denote the vector space underlying the corresponding representation of $\check{G}_{\mathbf{F}} = GL_n$. Consider the object

$$H((V \otimes \underline{V_0^*}) \boxtimes \cdots \boxtimes (V \otimes \underline{V_0^*}), \mathcal{S}_E) \in \mathrm{D}(\mathrm{Bun}_n \times X^d).$$

It is Σ_d-equivariant, and by assumption, it is isomorphic to the perverse sheaf

$$\mathcal{S}_E \boxtimes ((E \otimes \underline{V_0^*}[1]) \boxtimes \cdots \boxtimes (V \otimes \underline{V_0^*}[1])).$$

By restricting both sides to the diagonal $\mathrm{Bun}_n \times X \subset \mathrm{Bun}_n \times X^d$, we obtain a Σ_d-equivariant isomorphism

$$H((V \otimes \underline{V_0^*})^{\otimes d}, \mathcal{S}_E) \simeq \mathcal{S}_E \boxtimes (E \otimes \underline{V_0^*})^{\otimes d}[1]. \tag{7}$$

Moreover, both sides of (7) are acted on by the group $GL(\underline{V_0})$ of automorphisms of the vector space $\underline{V_0}$, and the isomorphism of (7) is compatible with these actions.

Let us take Σ_d-coinvariants of both sides of (7). By Lemma 6.5, we obtain a $GL(\underline{V_0})$-equivariant isomorphism of perverse sheaves:

$$H(\mathrm{Sym}^{*,d}(V \otimes \underline{V_0^*}), \mathcal{S}_E) \simeq \mathcal{S}_E \boxtimes \mathrm{Sym}^{*,d}(E \otimes \underline{V_0^*})[1]. \tag{8}$$

Let now $V \in \mathrm{Rep}(\check{G}_{\mathbf{F}})$ be a positive representation of degree d. Let $\underline{V}$ denote the underlining vector space, which we may regard as a representation of $GL(\underline{V_0})$. Note that we have an isomorphism of $\check{G}_{\mathbf{F}}$-representations

$$(\mathrm{Sym}^{*,d}(V \otimes \underline{V_0^*}) \otimes \underline{V})^{GL(\underline{V_0})} \simeq V.$$

Let us tensor both sides of (8) by $\underline{V}$ and take $GL(\underline{V_0})$-invariant parts. By Lemma 6.5, we obtain:

$$H(V, \mathcal{S}_E) \simeq \mathcal{S}_E \boxtimes (\mathrm{Sym}^{*,d}(E \otimes \underline{V_0^*}) \otimes \underline{V})^{GL(\underline{V_0})}[1] \simeq \mathcal{S}_E \boxtimes E^V[1],$$

where E^V is the local system corresponding to E and the $\check{G}_{\mathbf{F}}$-representation V.

Thus, we have constructed a functorial isomorphism $\alpha(V)$ for V positive of a fixed degree. To check the commutativity of the diagrams 1) and 2) it is sufficient to do this when $V_1 \simeq (V \otimes \underline{V_0^*})^{d_1}$ and $V_2 \simeq (V \otimes \underline{V_0^*})^{d_2}$. In this case, the required commutativity follows by construction.

7. Appendix B: Proof of Theorem 2.6

In this section we will show how to deduce Theorem 2.6 from Theorem 2.2. We will work with sheaves over any ring of coefficients (e.g., $\mathbf{F}$ or $\mathbf{O}$) and we will regard the Langlands dual group $\check{G}$ as a group-scheme over this ring. We shall denote by $\mathrm{Rep}(\check{G})$ the category of algebraic representations of $\check{G}$.

7.1. We shall first consider the case $d = 1$. By [11], Proposition 2.2, to any object $V \in \mathrm{Rep}(\check{G})$ we can attach a spherical perverse sheaf $\mathcal{T}_V$ on $\mathrm{Gr}_{G,X}$. Let R be the algebra of functions on G, viewed as an ind-object of $\mathrm{Rep}(\check{G})$ via the left action of $\check{G}$ on itself. Let us denote by $\mathcal{R}_X$ the corresponding ind-object of $P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$. The right action of $\check{G}$ on itself endows $\mathcal{R}_X$ with a $\check{G}$-action.

We define the functor F: $P^{\check{G},1} \to \mathrm{Ind}(P^{\mathcal{G}}(\mathrm{Gr}_{G,X}))$ by the formula

$$\mathcal{K} \mapsto (s^*(\mathcal{K}) \otimes \mathcal{R}_X[1])^{\check{G}},$$

where the superscript $\check{G}$ designates $\check{G}$-invariants. We shall denote by the same symbol the extension of this functor onto $\mathrm{Ind}(P^{\check{G},1})$.

Let $\mathbf{1}_{\mathrm{Gr}_{G,X}}$ denote the natural section $X \mapsto \mathrm{Gr}_{G,X}$. We define the functor G: $P^{\mathcal{G}}(\mathrm{Gr}_{G,X}) \to \mathrm{Ind}(P^{\check{G},1})$ by

$$\mathcal{T} \mapsto h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(\mathcal{R}_X \star \mathcal{T})),$$

where $\star$ is the convolution functor on $P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$, and $h^0(\cdot)$ designates the 0-th perverse cohomology. We will denote by the same symbol the extension of G onto $\mathrm{Ind}(P^{\mathcal{G}}(\mathrm{Gr}_{G,X}))$.

Proposition 7.2: *The functors* F *and* G *map* $P^{\check{G},1}$ *to* $P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$ *and* $P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$ *to* $P^{\check{G},1}$, *respectively, and define mutually inverse equivalences of categories.*

The rest of this subsection is devoted to the proof of this proposition.

Let us consider the composition $\mathsf{G} \circ \mathsf{F}$: $P^{\check{G},1} \to \mathrm{Ind}(P^{\check{G},1})$. Since the functor of convolution is exact, for $\mathcal{K} \in P^{\check{G},1}$ we have:

$$\mathcal{R}_X \star (s^*(\mathcal{K}) \otimes \mathcal{R}_X[1])^{\check{G}} \simeq (s^*(\mathcal{K})[1] \otimes (\mathcal{R}_X \star \mathcal{R}_X))^{\check{G}},$$

where $\check{G}$ acts on $\mathcal{R}_X \star \mathcal{R}_X$ via the second multiple.

However, $\mathcal{R}_X \star \mathcal{R}_X \simeq \mathcal{R}_X \otimes R$, with the diagonal action of $\check{G}$. Hence, we must calculate

$$h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(s^*(\mathcal{K} \otimes R)[1] \otimes \mathcal{R}_X)). \tag{9}$$

Lemma 7.3: *For any* $\mathcal{K}' \in \mathrm{Ind}(P^{\check{G},1})$ *we have a canonical isomorphism*

$$\mathcal{K}' \simeq h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(s^*(\mathcal{K}')[1] \otimes \mathcal{R}_X)).$$

Proof: The embedding of the trivial representation into R gives rise to a map

$$\mathbf{1}_{\mathrm{Gr}_{G,X}\,*}(\mathrm{Const}_X[1]) \to \mathcal{R}_X, \tag{10}$$

where Const_X denotes the constant sheaf on X. Hence, for any $\mathcal{K}'$ as above, we have a map

$$\mathcal{K}' \to \mathbf{1}^!_{\mathrm{Gr}_{G,X}}(s^*(\mathcal{K}')[1] \otimes \mathcal{R}_X),$$

and we must show that the LHS identifies with the maximal sub-object of the RHS, supported on the image of $\mathbf{1}_{\mathrm{Gr}_{G,X}}$.

Let us first assume that $\mathcal{K}'$ is lisse. Then $s^*(\mathcal{K}') \otimes \mathcal{R}_X \in \mathrm{Ind}(P^{\mathcal{G}}(\mathrm{Gr}_{G,X}))$ is ULA with respect to the projection s: $\mathrm{Gr}_{G,X} \to X$. Therefore, to prove our assertion, it would be sufficient to show that for some (or any) point $x \in X$,

$$\mathcal{K}'_x \to h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,x}}(\mathcal{K}'_x \otimes \mathcal{R})) \tag{11}$$

is an isomorphism, where $\mathcal{K}'_x$ denotes the fiber of $\mathcal{K}'$ at x, and $\mathbf{1}^!_{\mathrm{Gr}_{G,x}}$ is the embedding of the unit point into $\mathrm{Gr}_{G,x}$. However, the latter assertion follows from the equivalence $\mathrm{Rep}(\mathcal{G}) \simeq P^{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})$.

Let now $\mathcal{K}'$ be arbitrary. We can write

$$\mathcal{K}'_1 \to \mathcal{K}' \to j_*j^*(\mathcal{K}') \to \mathcal{K}'_2, \tag{12}$$

where j is the embedding of an open subset $X' \hookrightarrow X$, and $\mathcal{K}'_1, \mathcal{K}'_2$ are perverse sheaves supported on $X - X'$. By choosing X' to be sufficiently small, we can arrange that the restriction of $\mathcal{K}'$ to X' be lisse.

Since the functor $h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,x}}(\cdot))$ is left exact, it is sufficient to show that the map in the lemma is an isomorphism for $\mathcal{K}'_1$, $\mathcal{K}'_2$ and $j_*j^*(\mathcal{K}')$. The assertion concerning $\mathcal{K}'_1$ and $\mathcal{K}'_2$ is an immediate corollary of the equivalence $\mathrm{Rep}(\check{G}) \simeq P^{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})$. In addition, we have:

$$\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(s^*(j_*j^*(\mathcal{K}')) \otimes \mathcal{R}_X) \simeq j_*j^*(\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(s^*(\mathcal{K}') \otimes \mathcal{R}_X)),$$

and our assertion follows from the lisse case, considered above. ∎

Using the lemma, we obtain

$$\mathsf{G} \circ \mathsf{F}(\mathcal{K}) \simeq (\mathcal{K} \otimes R)^{\check{G}} \simeq \mathcal{K},$$

as required. Let us now consider the composition

$$\mathsf{F} \circ \mathsf{G}\colon P^{\mathcal{G}}(\mathrm{Gr}_{G,X}) \to \mathrm{Ind}(P^{\mathcal{G}}(\mathrm{Gr}_{G,X})).$$

Lemma 7.4: *For $\mathcal{T} \in P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$ there exists a canonical isomorphism*

$$s^*(h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(\mathcal{R}_X \star \mathcal{T}))[1]) \otimes \mathcal{R}_X \simeq \mathcal{R}_X \star \mathcal{T}.$$

Proof: Let us rewrite the LHS of the expression in the lemma as

$$\mathcal{R}_X \star \mathbf{1}_{\mathrm{Gr}_{G,X}*}(h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(\mathcal{R}_X \star \mathcal{T}))). \tag{13}$$

The natural map

$$\mathbf{1}_{\mathrm{Gr}_{G,X}*}(h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(\mathcal{R}_X \star \mathcal{T}))) \to \mathcal{R}_X \star \mathcal{T}$$

gives rise to a map from the expression in (13) to

$$\mathcal{R}_X \star (\mathcal{R}_X \star \mathcal{T}) \simeq (\mathcal{R}_X \star \mathcal{R}_X) \star \mathcal{T}.$$

The multiplication on R give rise to a map $\mathcal{R}_X \star \mathcal{R}_X \to \mathcal{R}_X$. Hence, by composing, we obtain a map

$$\mathcal{R}_X \star \mathbf{1}_{\mathrm{Gr}_{G,X}*}(h^0(\mathbf{1}^!_{\mathrm{Gr}_{G,X}}(\mathcal{R}_X \star \mathcal{T}))) \to \mathcal{R}_X \star \mathcal{T}.$$

To show that the latter map is an isomorphism, we proceed as in the proof of Lemma 7.3, by reducing the assertion to the case when $\mathcal{T}$ is ULA with respect to s: $\mathrm{Gr}_{G,X} \to X$. ∎

Thus,

$$\mathsf{F} \circ \mathsf{G}(\mathcal{T}) \simeq (\mathcal{R}_X \star \mathcal{T})^{\check{G}},$$

where $\check{G}$ acts on the convolution via its action on $\mathcal{R}_X$. The map (10) gives rise to a map

$$\mathcal{T} \to (\mathcal{R}_X \star \mathcal{T})^{\check{G}},$$

and repeating the argument used in the proofs of Lemmas 7.3 and 7.4, we show that the latter map is an isomorphism.

Thus, we obtain that the functors F and G induce mutually inverse equivalences of categories of $\mathrm{Ind}(P^{\mathcal{G}}(\mathrm{Gr}_{G,X}))$ and $\mathrm{Ind}(P^{\check{G},1})$. In particular, F sends $P^{\check{G},1}$ to $P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$ and G sends $P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$ to $P^{\check{G},1}$, and the resulting functors $P^{\check{G},1} \rightleftarrows P^{\mathcal{G}}(\mathrm{Gr}_{G,X})$ are also mutually inverse.

7.5. To treat the case of an arbitrary d, we will have to construct an object ${}^f\mathcal{R}^{(d)}$ in $P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$, which will play a role similar to that of $\mathcal{R}_X$. This construction is essentially borrowed from [2], Sect. 3.4.

First, let us recall the following construction from [11], Sect. 5. Let Δ denote the embedding of the diagonal $X \to X \times X$ and j be the embedding of the complement. Then given two objects $V, W \in \mathrm{Rep}(\check{G})$ there exists a canonical map

$$j_*j^*(\mathcal{T}_V \boxtimes \mathcal{T}_W) \to \Delta_*(\mathcal{T}_{V\otimes W}).$$

Applying this to $V = W = R$, and composing with the map $\mathcal{T}_{R\otimes R} \to \mathcal{T}_R$, corresponding to the algebra structure on R, we obtain a map

$$j_*j^*(\mathcal{R}_X \boxtimes \mathcal{R}_X) \to \Delta_*(\mathcal{R}_X). \tag{14}$$

For a positive integer d, let us denote by J^d the finite set $\{1, \ldots, d\}$, and for a surjection of finite sets $J^d \twoheadrightarrow I$, let Δ^I be the embedding of the corresponding diagonal $X^I \hookrightarrow X^d = X^{J^d}$. Let $\overset{o}{\Delta}{}^I$ be the locally closed embedding of the open subset $\overset{o}{X}{}^I \hookrightarrow X^{J^d}$, obtained by removing from X^I its diagonal divisor. Let us denote by Gr^I_G the corresponding version of the affine Grassmannian over X^I (obtained as a pull-back from $\mathrm{Gr}^{|I|}_G$ over $X^{(|I|)}$), and let $\overset{o}{\mathrm{Gr}}{}^I_G$ be its restriction to $\overset{o}{X}{}^I$. By a slight abuse of notation, we shall denote by Δ^I and $\overset{o}{\Delta}{}^I$ the embeddings of Gr^I_G and $\overset{o}{\mathrm{Gr}}{}^I_G$ into $\mathrm{Gr}^{J^d}_G$, respectively. The basic factorization property of the affine Grassmannian (cf. [11], Sect. 5) is that we have a canonical isomorphism

$$\overset{o}{\mathrm{Gr}}{}^I_G \simeq (\mathrm{Gr}_{G,X})^{\times I}|_{\overset{o}{X}{}^I}.$$

Consider the perverse sheaf $\mathcal{R}^{\boxtimes I}_X|_{\overset{o}{\mathrm{Gr}}{}^I_G}$, and let us denote by $\overset{o}{\mathcal{R}}{}^I_X$ its *-direct image under $\overset{o}{\Delta}{}^I\colon \overset{o}{\mathrm{Gr}}{}^I_G \to \mathrm{Gr}^{J^d}_G$. From (14) we obtain that for any surjection $I \twoheadrightarrow I'$ with $|I'| = |I| - 1$ there exists a naturally defined map

$$\overset{o}{\mathcal{R}}{}^I_X \to \overset{o}{\mathcal{R}}{}^{I'}_X.$$

By appropriately choosing the signs (cf. [2], 3.4.11), we obtain a complex of perverse sheaves on $\mathrm{Gr}^{J^d}_G$, which we will denote by $\mathfrak{C}^\bullet(\mathcal{R}^d_X)$, and whose k-th term is

$$\bigoplus_{I,|I|=d-k} \overset{o}{\mathcal{R}}{}^I_X.$$

By essentially repeating the proof of Lemma 2.4.12 of [2], we obtain the following:

LEMMA 7.6: *The complex $\mathfrak{C}^\bullet(\mathcal{R}^d_X)$ is acyclic of degree* 0*.*

Let us denote by ${}^f\mathcal{R}^d_X$ the 0-th cohomology of $\mathfrak{C}^\bullet(\mathcal{R}^d_X)$.

Remark: The perverse sheaf ${}^f\mathcal{R}^d_X$ has been introduced by A. Beilinson in the construction of automorphic sheaves using a "spectral projector".

7.7. We shall now construct a version of ${}^f\mathcal{R}^d_X$ that lives on Gr^d_G instead of $\mathrm{Gr}^{J^d}_G$, which we will denote by ${}^f\mathcal{R}^{(d)}_X$. Let us denote by sym_d the natural map $\mathrm{Gr}^{J^d}_G \to \mathrm{Gr}^d_G$. If we worked with sheaves with characteristic 0 coefficients, we would define ${}^f\mathcal{R}^{(d)}_X$ as Σ_d-invariants in $(sym_d)_!({}^f\mathcal{R}^d_X)$. In the case of arbitrary coefficients, we proceed as follows.

We define ${}^f\mathcal{R}^{(d)}_X$ as the kernel of the map

$$((\mathrm{sym}_d)!(\mathfrak{C}^0(\mathcal{R}^d_X)))_{\Sigma_d} \to ((\mathrm{sym}_d)!(\mathfrak{C}^1(\mathcal{R}^d_X)))_{\Sigma_d},$$

where the subscript "Σ_d" means Σ_d-coinvariants, taken in the category of perverse sheaves.

By construction, ${}^f\mathcal{R}^{(d)}_X$ has the following factorization property. For a partition $\overline{d} : d = d_1 + \cdots + d_m$, we have an isomorphism

$$(15)\qquad {}^f\mathcal{R}^{(d)}_X|_{X^{(\overline{d})}_{disj} \underset{X^{(d)}}{\times} \mathrm{Gr}^d_G} \simeq (\mathcal{R}^{(d_1)}_X \boxtimes \cdots \boxtimes \mathcal{R}^{(d_m)}_X)|_{X^{(\overline{d})}_{disj} \underset{X^{(d)}}{\times} (\mathrm{Gr}^{d_1}_G \times \cdots \times \mathrm{Gr}^{d_m}_G)},$$

under the identification

$$X^{(\overline{d})}_{disj} \underset{X^{(d)}}{\times} \mathrm{Gr}^d_G \simeq X^{(\overline{d})}_{disj} \underset{X^{(d)}}{\times} (\mathrm{Gr}^{d_1}_G \times \cdots \times \mathrm{Gr}^{d_m}_G).$$

Consider the complex $\mathfrak{C}^\bullet(\mathcal{R}^{(d)}_X)$, whose terms are given by

$$\mathfrak{C}^k(\mathcal{R}^{(d)}_X) := ((\mathrm{sym}_d)!(\mathfrak{C}^k(\mathcal{R}^d_X)))_{\Sigma_d}.$$

Note that the k-term is a *-extension of a complex on Gr^d_G, which is supported over the locus of codimension k in $X^{(d)}$, corresponding to the collision pattern of points.

Proposition 7.8: *The complex*

$$^f\mathcal{R}^{(d)}_X \to \mathfrak{C}^0(\mathcal{R}^{(d)}_X) \to \mathfrak{C}^1(\mathcal{R}^{(d)}_X) \to \cdots \to \mathfrak{C}^{d-1}(\mathcal{R}^{(d)}_X)$$

is acyclic.

Proof: The assertion is evidently true for $d = 2$ and we proceed by induction. By (15), we can assume that the complex in question is acyclic off the preimage of main diagonal $\Delta\colon X \hookrightarrow X^{(d)}$.

Since the last arrow $\mathfrak{C}^{d-2}(\mathcal{R}^{(d)}_X) \to \mathfrak{C}^{d-1}(\mathcal{R}^{(d)}_X)$ is surjective, the assertion of the proposition is equivalent to the fact that the map

$$h^{d-1}(\Delta^!(^f\mathcal{R}^{(d)}_X)) \to \mathcal{R}_X,$$

resulting from the above complex, is an isomorphism.

Consider the perverse sheaf $((\mathrm{sym}_d)_!(^f\mathcal{R}^d_X))_{\Sigma_d}$ on Gr^d_G, which maps naturally to $^f\mathcal{R}^{(d)}_X$. Both the kernel and the cokernel of this map are supported over the preimage of the diagonal divisor in X. Therefore, the map

$$h^{d-1}(\Delta^!((\mathrm{sym}_d)_!(^f\mathcal{R}^d_X))_{\Sigma_d}) \to h^{d-1}(\Delta^!(^f\mathcal{R}^{(d)}_X)) \tag{16}$$

is surjective.

However, it is easy to see that

$$h^{d-1}(\Delta^!((\mathrm{sym}_d)_!(^f\mathcal{R}^d_X))_{\Sigma_d}) \simeq \Delta_*(\mathcal{R}_X),$$

and the resulting composition

$$\mathcal{R}_X \to h^{d-1}(\Delta^!(^f\mathcal{R}^{(d)}_X)) \to \mathcal{R}_X$$

is the identity map. Hence, the map of (16) is an isomorphism. ∎

7.9. Let us introduce the hybrid category $\mathrm{Rep}(\check{G}, P^{\mathcal{G}^d}(\mathrm{Gr}^d_G))$, which consists of $\mathcal{G}^d$-equivariant ind-perverse sheaves $\mathcal{T}$ on Gr^d_G, such that for every partition $\overline{d} : d = d_1, \dots, d_n$ the pull-back of $\mathcal{T}$ to $X^{(\overline{d})}_{disj} \underset{X^{(d)}}{\times} (\mathrm{Gr}^{d_1}_G \times \cdots \times \mathrm{Gr}^{d_m}_G)$ carries an action of $\check{G}^{\times m}$, such that conditions, parallel to 1) and 2) in the definition of $P^{\check{G},d}$ hold. By construction, ${}^f\mathcal{R}^{(d)}_X$ is an object of $\mathrm{Rep}(\check{G}, P^{\mathcal{G}^d}(\mathrm{Gr}^d_G))$.

We define the functor

$$\mathcal{T}' \mapsto \mathcal{T}'^{\check{G}} \colon \mathrm{Rep}(\check{G}, P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)) \to \mathrm{Ind}(P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)),$$

which sends $\mathcal{T}'$ to its maximal sub-ind perverse sheaf, on which all the actions of $\check{G}^{\times m}$ are trivial.

For an object $\mathcal{K} \in P^{\check{G},d}$ consider

$$s^*(\mathcal{K}) \overset{!}{\otimes} {}^f\mathcal{R}_X[-d] \in \mathrm{Rep}(\check{G}, P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)),$$

where s denotes the projection $\mathrm{Gr}^d_G \to X^{(d)}$, and where $\overset{!}{\otimes}$ denotes the functor $\mathbb{D}(\mathbb{D}(\cdot) \otimes \mathbb{D}(\cdot))$.

We define the functor $\mathsf{F}^d \colon P^{\check{G},d} \to \mathrm{Ind}(P^{\mathcal{G}^d}(\mathrm{Gr}^d_G))$ by

$$\mathcal{T} \mapsto (s^*(\mathcal{K}) \overset{!}{\otimes} {}^f\mathcal{R}_X[-d])^{\check{G}}.$$

Let $\mathbf{1}_{\mathrm{Gr}^d_G}$ denote the unit section of Gr^d_G. If $\mathcal{T}'$ is an object of $\mathrm{Rep}(\check{G}, P^{\mathcal{G}^d}(\mathrm{Gr}^d_G))$, then $h^0(\mathbf{1}^!_{\mathrm{Gr}^d_G}(\mathcal{T}'))$ is naturally an object of $\mathrm{Ind}(P^{\check{G},d})$.

For $\mathcal{T} \in P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$, we can consider ${}^f\mathcal{R}^{(d)}_X \underset{\mathrm{Gr}^d_G}{\star} \mathcal{T}$ as an object of $\mathrm{Rep}(\check{G}, P^{\mathcal{G}^d}(\mathrm{Gr}^d_G))$, where $\underset{\mathrm{Gr}^d_G}{\star}$ refers to the convolution on $P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$ (as opposed to the one, involving $d = d_1 + d_2$, as was the case in Sect. 2).

We define the functor $\mathsf{G}^d \colon P^{\mathcal{G}^d}(\mathrm{Gr}^d_G) \to \mathrm{Ind}(P^{\check{G},d})$ by

$$\mathcal{T} \mapsto h^0(\mathbf{1}^!_{\mathrm{Gr}^d_G}({}^f\mathcal{R}^{(d)}_X \star \mathcal{T})).$$

Proposition 7.10: *The functors F^d and G^d map to $P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$ and $P^{\check{G},d}$, respectively, and are mutually inverse equivalences of categories.*

The proof of this proposition essentially repeats that of Proposition 7.2, using the properties of ${}^f\mathcal{R}^{(d)}_X$ established in the previous subsection. This establishes the equivalence $P^{\check{G},d} \simeq P^{\mathcal{G}^d}(\mathrm{Gr}^d_G)$, stated in Theorem 2.6. It remains to show the compatibility of this equivalence and the $\star$ operations in (2) and (3).

For $d = d_1 + d_2$, we have a natural morphism ${}^f\mathcal{R}_X^{(d_1)} \star {}^f\mathcal{R}_X^{(d_2)} \to {}^f\mathcal{R}_X^{(d)}$. It gives rise to a functorial morphism, defined for $\mathcal{K}_1 \in P^{\check{G},d_1}$, $\mathcal{K}_2 \in P^{\check{G},d_2}$:

$$\mathsf{F}^{d_1}(\mathcal{K}_1) \star \mathsf{F}^{d_2}(\mathcal{K}_2) \to \mathsf{F}^d(\mathcal{K}_1 \star \mathcal{K}_2).$$

To show that the latter morphism is an isomorphism, we decompose Gr_G^d with respect to the diagonal stratification of $X^{(d)}$, and the assertion follows from the fact that the isomorphism of Proposition 7.2 intertwines the convolution of $\mathcal{G}$-equivariant perverse sheaves with tensor products of objects of $P^{\check{G},1}$.

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